Trigonometrical Identities

Exercise 21A

Question 1.

Prove:
$$\frac{\sec A - 1}{\sec A + 1} = \frac{1 - \cos A}{1 + \cos A}$$

Solution:

$$\text{LHS} = \frac{\sec A - 1}{\sec A + 1} = \frac{\frac{1}{\cos A} - 1}{\frac{1}{\cos A} + 1}$$

$$= \frac{1 - \cos A}{1 + \cos A} = \text{RHS}$$

Question 2.

Prove:
$$\frac{1 + \sin A}{1 - \sin A} = \frac{\operatorname{cosec} A + 1}{\operatorname{cosec} A - 1}$$

Solution:

$$\text{LHS} = \frac{1 + \sin A}{1 - \sin A}$$

$$\text{RHS} = \frac{\operatorname{cosec} A + 1}{\operatorname{cosec} A - 1} = \frac{\frac{1}{\sin A} + 1}{\frac{1}{\sin A} - 1}$$

$$= \frac{1 + \sin A}{1 - \sin A}$$

Question 3.

Prove:
$$\frac{1}{\tan A + \cot A} = \sin A \cos A$$

Solution:

$$\frac{1}{\tan A + \cot A} = \sin A \cos A$$

$$\text{LHS} = \frac{1}{\tan A + \cot A}$$

$$= \frac{1}{\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}} = \frac{1}{\frac{\sin^2 A + \cos^2 A}{\sin A \cos A}}$$

$$= \frac{1}{\frac{1}{\sin A \cos A}} \left(\because \sin^2 A + \cos^2 A = 1\right)$$

$$= \sin A \cos A = \text{RHS}$$

Question 4.

Prove:

$$\tan A - \cot A = \frac{1 - 2\cos^2 A}{\sin A \cos A}$$

Solution:

$$\tan A - \cot A = \frac{\sin A}{\cos A} - \frac{\cos A}{\sin A}$$

$$= \frac{\sin^2 A - \cos^2 A}{\sin A \cos A}$$

$$= \frac{1 - \cos^2 A - \cos^2 A}{\sin A \cos A} \left(\because \sin^2 A = 1 - \cos^2 A\right)$$

$$= \frac{1 - 2\cos^2 A}{\sin A \cos A}$$

Question 5.

Prove:

$$\sin^4 A - \cos^4 A = 2\sin^2 A - 1$$

Solution:

$$\sin^4 A - \cos^4 A$$

$$= (\sin^2 A)^2 - (\cos^2 A)^2$$

$$= (\sin^2 A + \cos^2 A)(\sin^2 A - \cos^2 A)$$

$$= \sin^2 A - \cos^2 A$$

$$= \sin^2 A - (1 - \sin^2 A)$$

$$= 2\sin^2 A - 1$$

Question 6.

Prove:

$$(1-\tan A)^2+(1+\tan A)^2=2\sec^2 A$$

Solution:

$$(1-\tan A)^2+(1+\tan A)^2$$
$$=\left(1+\tan^2 A-2\tan A\right)+\left(1+\tan^2 A+2\tan A\right)$$
$$=2(1+\tan^2 A)$$
$$=2\sec^2 A$$

Question 7.

Prove:

$$\operatorname{cosec}^4 A-\operatorname{cosec}^2 A=\cot^4 A+\cot^2 A$$

Solution:

$$\text{LHS}=\operatorname{cosec}^4 A-\operatorname{cosec}^2 A$$
$$=\operatorname{cosec}^2 A(\operatorname{cosec}^2 A-1)$$
$$\text{RHS}=\cot^4 A+\cot^2 A$$
$$=\cot^2 A(\cot^2 A+1)$$
$$=(\operatorname{cosec}^2 A-1)\operatorname{cosec}^2 A$$
$$\text{Thus, LHS}=\text{RHS}$$

Question 8.

Prove:

$$\sec A(1-\sin A)(\sec A+\tan A)=1$$

Solution:

$$\text{LHS}=\sec A(1-\sin A)(\sec A+\tan A)$$
$$=\frac{1}{\cos A}(1-\sin A)\left(\frac{1}{\cos A}+\frac{\sin A}{\cos A}\right)$$
$$=\frac{(1-\sin A)}{\cos A}\left(\frac{1+\sin A}{\cos A}\right)=\left(\frac{1-\sin^2 A}{\cos^2 A}\right)$$
$$=\left(\frac{\cos^2 A}{\cos^2 A}\right)=1=\text{RHS}$$

Question 9.

Prove:

$\operatorname{cosec} A(1+\cos A)(\operatorname{cosec} A-\cot A)=1$

Solution:

$$\text{LHS}=\operatorname{cosec} A(1+\cos A)(\operatorname{cosec} A-\cot A)$$
$$=\frac{1}{\sin A}(1+\cos A)\left(\frac{1}{\sin A}-\frac{\cos A}{\sin A}\right)$$
$$=\frac{(1+\cos A)}{\sin A}\left(\frac{1-\cos A}{\sin A}\right)$$
$$=\frac{1-\cos^2 A}{\sin^2 A}=\frac{\sin^2 A}{\sin^2 A}=1=\text{RHS}$$

Question 10.

Prove:

$\sec^2 A+\operatorname{cosec}^2 A=\sec^2 A\operatorname{cosec}^2 A$

Solution:

$$\text{LHS}=\sec^2 A+\operatorname{cosec}^2 A$$
$$=\frac{1}{\cos^2 A}+\frac{1}{\sin^2 A}=\frac{\sin^2 A+\cos^2 A}{\cos^2 A.\sin^2 A}$$
$$=\frac{1}{\cos^2 A.\sin^2 A}=\sec^2 A\operatorname{cosec}^2 A=\text{RHS}$$

Question 11.

Prove:

$$\frac{\left(1+\tan^2 A\right)\cot A}{\operatorname{cosec}^2 A}=\tan A$$

Solution:

$$\frac{\left(1+\tan^2 A\right)\cot A}{\operatorname{cosec}^2 A}$$

$$=\frac{\sec^2 A\cot A}{\operatorname{cosec}^2 A}\left(\because \sec^2 A = 1+\tan^2 A\right)$$

$$=\frac{\frac{1}{\cos^2 A}\times\frac{\cos A}{\sin A}}{\frac{1}{\sin^2 A}}=\frac{\frac{1}{\cos A\sin A}}{\frac{1}{\sin^2 A}}$$

$$=\frac{\sin A}{\cos A}=\tan A$$

Question 12.

Prove:

$\tan^2 A - \sin^2 A = \tan^2 A.\sin^2 A$

Solution:

$$\text{LHS} = \tan^2 A - \sin^2 A$$

$$=\frac{\sin^2 A}{\cos^2 A}-\sin^2 A=\frac{\sin^2 A(1-\cos^2 A)}{\cos^2 A}$$

$$=\frac{\sin^2 A}{\cos^2 A}.\sin^2 A=\tan^2 A.\sin^2 A=\text{RHS}$$

Question 13.

Prove:

$\cot^2 A - \cos^2 A = \cos^2 A.\cot^2 A$

Solution:

$$\text{LHS} = \cot^2 A - \cos^2 A$$

$$=\frac{\cos^2 A}{\sin^2 A}-\cos^2 A=\frac{\cos^2 A(1-\sin^2 A)}{\sin^2 A}$$

$$=\cos^2 A\frac{\cos^2 A}{\sin^2 A}=\cos^2 A.\cot^2 A=\text{RHS}$$

Question 14.

Prove:

$(\operatorname{cosec}A + \sin A)(\operatorname{cosec}A - \sin A) = \cot^2 A + \cos^2 A$

Solution:

$(\operatorname{cosec}A + \sin A)(\operatorname{cosec}A - \sin A)$

$= \operatorname{cosec}^2 A - \sin^2 A$

$= (1 + \cot^2 A) - (1 - \cos^2 A)$

$= \cot^2 A + \cos^2 A$

Question 15.

Prove:

$(\sec A - \cos A)(\sec A + \cos A) = \sin^2 A + \tan^2 A$

Solution:

$(\sec A - \cos A)(\sec A + \cos A)$

$= \sec^2 A - \cos^2 A$

$= (1 + \tan^2 A) - (1 - \sin^2 A)$

$= \sin^2 A + \tan^2 A$

Question 16.

Prove:

$(\cos A + \sin A)^2 + (\cos A - \sin A)^2 = 2$

Solution:

$\text{LHS} = (\cos A + \sin A)^2 + (\cos A - \sin A)^2$

$= \cos^2 A + \sin^2 A + 2\cos A.\sin A + \cos^2 A + \sin^2 A - 2\cos A.\sin A$

$= 2(\cos^2 A + \sin^2 A) = 2 = \text{RHS}$

Question 17.

Prove:

$(\operatorname{cosec}A - \sin A)(\sec A - \cos A)(\tan A + \cot A) = 1$

Solution:

$$LHS = (\operatorname{cosec}A - \sin A)(\sec A - \cos A)(\tan A + \cot A)$$
$$= \left(\frac{1}{\sin A} - \sin A\right)\left(\frac{1}{\cos A} - \cos A\right)\left(\tan A + \frac{1}{\tan A}\right)$$
$$= \left(\frac{1 - \sin^2 A}{\sin A}\right)\left(\frac{1 - \cos^2 A}{\cos A}\right)\left(\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}\right)$$
$$= \left(\frac{\cos^2 A}{\sin A}\right)\left(\frac{\sin^2 A}{\cos A}\right)\left(\frac{\sin^2 A + \cos^2 A}{\sin A.\cos A}\right)$$
$$= 1$$

Question 18.

Prove:

$$\frac{1}{\sec A + \tan A} = \sec A - \tan A$$

Solution:

$$\frac{1}{\sec A + \tan A}$$
$$= \frac{1}{\sec A + \tan A} \times \frac{\sec A - \tan A}{\sec A - \tan A}$$
$$= \frac{\sec A - \tan A}{\sec^2 A - \tan^2 A}$$
$$= \sec A - \tan A$$

Question 19.

Prove:

$$\operatorname{cosec}A + \cot A = \frac{1}{\operatorname{cosec}A - \cot A}$$

Solution:

$$\operatorname{cosec}A + \cot A$$
$$= \frac{\operatorname{cosec}A + \cot A}{1} \times \frac{\operatorname{cosec}A - \cot A}{\operatorname{cosec}A - \cot A}$$
$$= \frac{\operatorname{cosec}^2 A - \cot^2 A}{\operatorname{cosec}A - \cot A} = \frac{1 + \cot^2 A - \cot^2 A}{\operatorname{cosec}A - \cot A}$$
$$= \frac{1}{\operatorname{cosec}A - \cot A}$$

Question 20.

Prove:

$$\frac{\sec A - \tan A}{\sec A + \tan A} = 1 - 2\sec A \tan A + 2\tan^2 A$$

Solution:

$$\frac{\sec A - \tan A}{\sec A + \tan A}$$

$$= \frac{\sec A - \tan A}{\sec A + \tan A} \times \frac{\sec A - \tan A}{\sec A - \tan A}$$

$$= \frac{(\sec A - \tan A)^2}{\sec^2 A - \tan^2 A}$$

$$= \frac{\sec^2 A + \tan^2 A - 2\sec A \tan A}{1}$$

$$= 1 + \tan^2 A + \tan^2 A - 2\sec A \tan A$$

$$= 1 - 2\sec A \tan A + 2\tan^2 A$$

Question 21.

Prove:

$$(\sin A + cosecA)^2 + (\cos A + \sec A)^2 = 7 + \tan^2 A + \cot^2 A$$

Solution:

$$(\sin A + cosecA)^2 + (\cos A + \sec A)^2$$

$$= \sin^2 A + cosec^2 A + 2\sin A \, cosecA + \cos^2 A + \sec^2 A + 2\cos A \sec A$$

$$= \sin^2 A + \cos^2 A + cosec^2 A + \sec^2 A + 2 + 2$$

$$= 1 + cosec^2 A + \sec^2 A + 4$$

$$= (1 + \cot^2 A) + (1 + \tan^2 A) + 5$$

$$= 7 + \tan^2 A + \cot^2 A$$

Question 22.

Prove:

$$\sec^2 A . cosec^2 A = \tan^2 A + \cot^2 A + 2$$

Solution:

$$\text{LHS} = \sec^2 A.\operatorname{cosec}^2 A = \frac{1}{\cos^2 A.\sin^2 A}$$

$$\text{RHS} = \tan^2 A + \cot^2 A + 2 = \tan^2 A + \cot^2 A + 2\tan A.\cot A$$

$$= (\tan A + \cot A)^2 = \left(\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}\right)^2$$

$$= \left(\frac{\sin^2 A + \cos^2 A}{\sin A.\cos A}\right)^2 = \frac{1}{\cos^2 A.\sin^2 A}$$

Hence, LHS = RHS

Question 23.

Prove:

$$\frac{1}{1+\cos A} + \frac{1}{1-\cos A} = 2\operatorname{cosec}^2 A$$

Solution:

$$\frac{1}{1+\cos A} + \frac{1}{1-\cos A}$$

$$= \frac{1-\cos A + 1 + \cos A}{(1+\cos A)(1-\cos A)}$$

$$= \frac{2}{1-\cos^2 A}$$

$$= \frac{2}{\sin^2 A}$$

$$= 2\operatorname{cosec}^2 A$$

Question 24.

Prove:

$$\frac{1}{1-\sin A} + \frac{1}{1+\sin A} = 2\sec^2 A$$

Solution:

$$\frac{1}{1-\sin A} + \frac{1}{1+\sin A}$$

$$= \frac{1+\sin A + 1 - \sin A}{(1-\sin A)(1+\sin A)}$$

$$= \frac{2}{1-\sin^2 A}$$

$= \frac{2}{\cos^2 A}$

$= 2\sec^2 A$

Question 25.

Prove:

$$\frac{\operatorname{cosec} A}{\operatorname{cosec} A - 1} + \frac{\operatorname{cosec} A}{\operatorname{cosec} A + 1} = 2\sec^2 A$$

Solution:

$$\frac{\operatorname{cosec} A}{\operatorname{cosec} A - 1} + \frac{\operatorname{cosec} A}{\operatorname{cosec} A + 1}$$

$$= \frac{\operatorname{cosec}^2 A + \operatorname{cosec} A + \operatorname{cosec}^2 A - \operatorname{cosec} A}{\operatorname{cosec}^2 A - 1}$$

$$= \frac{2\operatorname{cosec}^2 A}{\cot^2 A} \left(\because \operatorname{cosec}^2 A - 1 = \cot^2 A\right)$$

$$= \frac{\frac{2}{\sin^2 A}}{\frac{\cos^2 A}{\sin^2 A}} = \frac{2}{\cos^2 A} = 2\sec^2 A$$

Question 26.

Prove:

$$\frac{\sec A}{\sec A + 1} + \frac{\sec A}{\sec A - 1} = 2\operatorname{cosec}^2 A$$

Solution:

$$\frac{\sec A}{\sec A + 1} + \frac{\sec A}{\sec A - 1}$$

$$= \frac{\sec^2 A - \sec A + \sec^2 A + \sec A}{\sec^2 A - 1}$$

$$= \frac{2\sec^2 A}{\tan^2 A} \left(\because \sec^2 A - 1 = \tan^2 A\right)$$

$$= \frac{\frac{2}{\cos^2 A}}{\frac{\sin^2 A}{\cos^2 A}} = \frac{2}{\sin^2 A} = 2\operatorname{cosec}^2 A$$

Question 27.

Prove:

$$\frac{1+\cos A}{1-\cos A}=\frac{\tan^2 A}{(\sec A-1)^2}$$

Solution:

$$\frac{1+\cos A}{1-\cos A}$$

$$=\frac{1+\dfrac{1}{\sec A}}{1-\dfrac{1}{\sec A}}=\frac{\sec A+1}{\sec A-1}$$

$$=\frac{\sec A+1}{\sec A-1}\times\frac{\sec A-1}{\sec A-1}$$

$$=\frac{\sec^2 A-1}{(\sec A-1)^2}=\frac{\tan^2 A}{(\sec A-1)^2}\ (\because \sec^2 A-1=\tan^2 A)$$

Question 28.

Prove:

$$\frac{\cot^2 A}{(\operatorname{cosec} A+1)^2}=\frac{1-\sin A}{1+\sin A}$$

Solution:

$$\text{R.H.S}=\frac{1-\sin A}{1+\sin A}$$

$$=\frac{1-\dfrac{1}{\operatorname{cosec} A}}{1+\dfrac{1}{\operatorname{cosec} A}}=\frac{\operatorname{cosec} A-1}{\operatorname{cosec} A+1}$$

$$=\frac{\operatorname{cosec} A-1}{\operatorname{cosec} A+1}\times\frac{\operatorname{cosec} A+1}{\operatorname{cosec} A+1}$$

$$=\frac{\operatorname{cosec}^2 A-1}{(\operatorname{cosec} A+1)^2}=\frac{\cot^2 A}{(\operatorname{cosec} A+1)^2}\ (\because \operatorname{cosec}^2 A-1=\cot^2 A)$$

$$=\text{L.H.S}$$

Question 29.

Prove:

$$\frac{1+\sin A}{\cos A}+\frac{\cos A}{1+\sin A}=2\sec A$$

Solution:

$$\frac{1+\sin A}{\cos A}+\frac{\cos A}{1+\sin A}$$
$$=\frac{(1+\sin A)^2+\cos^2 A}{\cos A(1+\sin A)}$$
$$=\frac{1+\sin^2 A+2\sin A+\cos^2 A}{\cos A(1+\sin A)}$$
$$=\frac{1+2\sin A+1}{\cos A(1+\sin A)}$$
$$=\frac{2(1+\sin A)}{\cos A(1+\sin A)}$$
$$=2\sec A$$

Question 30.

Prove:

$$\frac{1-\sin A}{1+\sin A}=(\sec A-\tan A)^2$$

Solution:

$$\frac{1-\sin A}{1+\sin A}$$
$$=\frac{1-\sin A}{1+\sin A}\times\frac{1-\sin A}{1-\sin A}$$
$$=\frac{(1-\sin A)^2}{1-\sin^2 A}$$
$$=\frac{(1-\sin A)^2}{\cos^2 A}$$
$$=\left(\frac{1-\sin A}{\cos A}\right)^2$$
$$=(\sec A-\tan A)^2$$

Question 31.

Prove:

$$(\cot A - \operatorname{cosec} A)^2 = \frac{1-\cos A}{1+\cos A}$$

Solution:

$$R.H.S. = \frac{1-\cos A}{1+\cos A}$$
$$= \frac{1-\cos A}{1+\cos A} \times \frac{1-\cos A}{1-\cos A}$$
$$= \frac{(1-\cos A)^2}{1-\cos^2 A}$$
$$= \frac{(1-\cos A)^2}{\sin^2 A}$$
$$= \left(\frac{1-\cos A}{\sin A}\right)^2$$
$$= (\operatorname{cosec} A - \cot A)^2$$
$$= (\cot A - \operatorname{cosec} A)^2$$
$$= L.H.S$$

Question 32.

Prove:

$$\frac{\operatorname{cosec} A - 1}{\operatorname{cosec} A + 1} = \left(\frac{\cos A}{1+\sin A}\right)^2$$

Solution:

$$\frac{\operatorname{cosec} A - 1}{\operatorname{cosec} A + 1}$$
$$= \frac{\operatorname{cosec} A - 1}{\operatorname{cosec} A + 1} \times \frac{\operatorname{cosec} A + 1}{\operatorname{cosec} A + 1}$$
$$= \frac{\operatorname{cosec}^2 A - 1}{(\operatorname{cosec} A + 1)^2}$$
$$= \frac{\cot^2 A}{(\operatorname{cosec} A + 1)^2}$$

$$= \frac{\frac{\cos^2 A}{\sin^2 A}}{\left(\frac{1}{\sin A} + 1\right)^2}$$

$$= \left(\frac{\cos A}{1 + \sin A}\right)^2$$

Question 33.

Prove:

$$\tan^2 A - \tan^2 B = \frac{\sin^2 A - \sin^2 B}{\cos^2 A \cos^2 B}$$

Solution:

$$\tan^2 A - \tan^2 B$$

$$= \frac{\sin^2 A}{\cos^2 A} - \frac{\sin^2 B}{\cos^2 B}$$

$$= \frac{\sin^2 A \cos^2 B - \sin^2 B \cos^2 A}{\cos^2 A \cos^2 B}$$

$$= \frac{\sin^2 A(1 - \sin^2 B) - \sin^2 B(1 - \sin^2 A)}{\cos^2 A \cos^2 B}$$

$$= \frac{\sin^2 A - \sin^2 A \sin^2 B - \sin^2 B + \sin^2 A \sin^2 B}{\cos^2 A \cos^2 B}$$

$$= \frac{\sin^2 A - \sin^2 B}{\cos^2 A \cos^2 B}$$

Question 34.

Prove:

$$\frac{\sin A - 2\sin^3 A}{2\cos^3 A - \cos A} = \tan A$$

Solution:

$$\frac{\sin A - 2\sin^3 A}{2\cos^3 A - \cos A}$$

$$= \frac{\sin A(1 - 2\sin^2 A)}{\cos A(2\cos^2 A - 1)}$$

$$= \frac{\sin A(\sin^2 A + \cos^2 A - 2\sin^2 A)}{\cos A(2\cos^2 A - \sin^2 A - \cos^2 A)}$$

$$= \frac{\sin A(\cos^2 A - \sin^2 A)}{\cos A(\cos^2 A - \sin^2 A)}$$

$$= \frac{\sin A}{\cos A}$$

$$= \tan A$$

Question 35.

Prove:

$$\frac{\sin A}{1 + \cos A} = \operatorname{cosec} A - \cot A$$

Solution:

$$\frac{\sin A}{1 + \cos A}$$

$$= \frac{\sin A}{1 + \cos A} \times \frac{1 - \cos A}{1 - \cos A}$$

$$= \frac{\sin A(1 - \cos A)}{1 - \cos^2 A}$$

$$= \frac{\sin A(1 - \cos A)}{\sin^2 A}$$

$$= \frac{1 - \cos A}{\sin A}$$

$$= \frac{1}{\sin A} - \frac{\cos A}{\sin A}$$

$$= \operatorname{cosec} A - \cot A$$

Question 36.

Prove:

$$\frac{\cos A}{1 - \sin A} = \sec A + \tan A$$

Solution:

$$\text{LHS} = \frac{\cos A}{1-\sin A}$$

$$\text{RHS} = \sec A + \tan A$$

$$= \frac{1}{\cos A} + \frac{\sin A}{\cos A} = \frac{1+\sin A}{\cos A}$$

$$= \frac{1+\sin A}{\cos A}\left(\frac{1-\sin A}{1-\sin A}\right) = \left(\frac{1-\sin^2 A}{\cos A(1-\sin A)}\right)$$

$$= \frac{\cos^2 A}{\cos A(1-\sin A)} = \frac{\cos A}{(1-\sin A)} = \text{LHS}$$

Question 37.

Prove:

$$\frac{\sin A \tan A}{1-\cos A} = 1 + \sec A$$

Solution:

$$\frac{\sin A \tan A}{1-\cos A}$$

$$= \frac{\sin A \tan A}{1-\cos A} \times \frac{1+\cos A}{1+\cos A}$$

$$= \frac{\sin A \tan A(1+\cos A)}{1-\cos^2 A}$$

$$= \frac{\sin A \dfrac{\sin A}{\cos A}(1+\cos A)}{\sin^2 A}$$

$$= \frac{1+\cos A}{\cos A}$$

$$= \frac{1}{\cos A} + \frac{\cos A}{\cos A}$$

$$= \sec A + 1$$

Question 38.

Prove: $(1 + \cot A - \operatorname{cosec} A)(1 + \tan A + \sec A) = 2$

Solution:

$$(1 + \cot A - \operatorname{cosec} A)(1 + \tan A + \sec A)$$

$$= \left(1 + \frac{\cos A}{\sin A} - \frac{1}{\sin A}\right)\left(1 + \frac{\sin A}{\cos A} + \frac{1}{\cos A}\right)$$

$$= \left(\frac{\sin A + \cos A - 1}{\sin A}\right)\left(\frac{\cos A + \sin A + 1}{\cos A}\right)$$

$$= \frac{(\sin A + \cos A - 1)(\sin A + \cos A + 1)}{\sin A \cos A}$$

$$= \frac{(\sin A + \cos A)^2 - (1)^2}{\sin A \cos A}$$

$$= \frac{\sin^2 A + \cos^2 A + 2\sin A \cos A - 1}{\sin A \cos A}$$

$$= \frac{1 + 2\sin A \cos A - 1}{\sin A \cos A}$$

$$= \frac{2\sin A \cos A}{\sin A \cos A} = 2$$

Question 39.

Prove:

$$\sqrt{\frac{1+\sin A}{1-\sin A}} = \sec A + \tan A$$

Solution:

$$\sqrt{\frac{1+\sin A}{1-\sin A}}$$

$$= \sqrt{\frac{1+\sin A}{1-\sin A} \times \frac{1+\sin A}{1+\sin A}}$$

$$= \sqrt{\frac{(1+\sin A)^2}{1-\sin^2 A}} = \sqrt{\frac{(1+\sin A)^2}{\cos^2 A}}$$

$$= \frac{1+\sin A}{\cos A}$$

$$= \sec A + \tan A$$

Question 40.

Prove:

$$\sqrt{\frac{1-\cos A}{1+\cos A}} = \operatorname{cosec} A - \cot A$$

Solution:

$$\sqrt{\frac{1-\cos A}{1+\cos A}}$$
$$= \sqrt{\frac{1-\cos A}{1+\cos A} \times \frac{1-\cos A}{1-\cos A}}$$
$$= \sqrt{\frac{(1-\cos A)^2}{1-\cos^2 A}}$$
$$= \sqrt{\frac{(1-\cos A)^2}{\sin^2 A}}$$
$$= \frac{1-\cos A}{\sin A}$$
$$= \operatorname{cosec} A - \cot A$$

Question 41.

Prove:

$$\sqrt{\frac{1-\cos A}{1+\cos A}} = \frac{\sin A}{1+\cos A}$$

Solution:

$$\sqrt{\frac{1-\cos A}{1+\cos A}}$$
$$= \sqrt{\frac{1-\cos A}{1+\cos A} \times \frac{1+\cos A}{1+\cos A}}$$
$$= \sqrt{\frac{1-\cos^2 A}{(1+\cos A)^2}}$$
$$= \sqrt{\frac{\sin^2 A}{(1+\cos A)^2}}$$
$$= \frac{\sin A}{1+\cos A}$$

Question 42.

Prove:

$$\sqrt{\frac{1-\sin A}{1+\sin A}}=\frac{\cos A}{1+\sin A}$$

Solution:

$$\sqrt{\frac{1-\sin A}{1+\sin A}}$$

$$=\sqrt{\frac{1-\sin A}{1+\sin A}\times\frac{1+\sin A}{1+\sin A}}$$

$$=\sqrt{\frac{1-\sin^2 A}{(1+\sin A)^2}}$$

$$=\sqrt{\frac{\cos^2 A}{(1+\sin A)^2}}$$

$$=\frac{\cos A}{1+\sin A}$$

Question 43.

Prove:

$$1-\frac{\cos^2 A}{1+\sin A}=\sin A$$

Solution:

$$1-\frac{\cos^2 A}{1+\sin A}$$

$$=\frac{1+\sin A-\cos^2 A}{1+\sin A}$$

$$=\frac{\sin A+\sin^2 A}{1+\sin A}$$

$$=\frac{\sin A(1+\sin A)}{1+\sin A}$$

$$=\sin A$$

Question 44.

Prove:

$$\frac{1}{\sin A+\cos A}+\frac{1}{\sin A-\cos A}=\frac{2\sin A}{1-2\cos^2 A}$$

Solution:

$$\frac{1}{\sin A+\cos A}+\frac{1}{\sin A-\cos A}$$

$$=\frac{\sin A-\cos A+\sin A+\cos A}{\sin^2 A-\cos^2 A}$$

$$=\frac{2\sin A}{1-\cos^2 A-\cos^2 A}=\frac{2\sin A}{1-2\cos^2 A}$$

Question 45.

Prove:

$$\frac{\sin A+\cos A}{\sin A-\cos A}+\frac{\sin A-\cos A}{\sin A+\cos A}=\frac{2}{2\sin^2 A-1}$$

Solution:

$$\frac{\sin A+\cos A}{\sin A-\cos A}+\frac{\sin A-\cos A}{\sin A+\cos A}$$

$$=\frac{(\sin A+\cos A)^2+(\sin A-\cos A)^2}{(\sin A+\cos A)(\sin A-\cos A)}$$

$$=\frac{\sin^2 A+\cos^2 A+2\sin A\cos A+\sin^2 A+\cos^2 A-2\sin\cos A}{\sin^2 A-\cos^2 A}$$

$$=\frac{2\left(\sin^2 A+\cos^2 A\right)}{\sin^2 A-\cos^2 A}$$

$$=\frac{2}{\sin^2 A-\cos^2 A} \qquad [\sin^2 A+\cos^2 A=1]$$

$$=\frac{2}{\sin^2 A-\cos^2 A}=\frac{2}{\sin^2 A-(1-\sin^2 A)}$$

$$\Rightarrow \frac{2}{2\sin^2 A-1}$$

Question 46.

Prove:

$$\frac{\cot A + \operatorname{cosec} A - 1}{\cot A - \operatorname{cosec} A + 1} = \frac{1 + \cos A}{\sin A}$$

Solution:

$$\frac{\cot A + \operatorname{cosec} A - 1}{\cot A - \operatorname{cosec} A + 1}$$

$$= \frac{\cot A + \operatorname{cosec} A - (\operatorname{cosec}^2 A - \cot^2 A)}{\cot A - \operatorname{cosec} A + 1} \qquad [\operatorname{cosec}^2 A - \cot^2 A = 1]$$

$$= \frac{\cot A + \operatorname{cosec} A - [(\operatorname{cosec} A - \cot A)(\operatorname{cosec} A + \cot A)]}{\cot A - \operatorname{cosec} A + 1}$$

$$= \frac{\cot A + \operatorname{cosec} A[1 - \operatorname{cosec} A + \cot A]}{\cot A - \operatorname{cosec} A + 1}$$

$$= \cot A + \operatorname{cosec} A$$

$$= \frac{\cos A}{\sin A} + \frac{1}{\sin A}$$

$$= \frac{1 + \cos A}{\sin A}$$

Question 47.

Prove:

$$\frac{\sin\theta \tan\theta}{1 - \cos\theta} = 1 + \sec\theta$$

Solution:

$$\frac{\sin\theta \tan\theta}{1 - \cos\theta}$$

$$= \frac{\sin\theta \tan\theta}{1 - \cos\theta} \times \frac{1 + \cos\theta}{1 + \cos\theta}$$

$$= \frac{\sin\theta \tan\theta(1 + \cos\theta)}{1 - \cos^2\theta}$$

$$= \frac{\sin\theta \frac{\sin\theta}{\cos\theta}(1 + \cos\theta)}{\sin^2\theta}$$

$$= \frac{(1 + \cos\theta)}{\cos\theta}$$

$$= \frac{1}{\cos\theta} + 1$$

$$= \sec\theta + 1$$

Question 48.

Prove:

$$\frac{\cos\theta\cot\theta}{1+\sin\theta} = \cos ec\theta - 1$$

Solution:

$$\frac{\cos\theta\cot\theta}{1+\sin\theta}$$
$$= \frac{\cos\theta\cot\theta}{1+\sin\theta} \times \frac{1-\sin\theta}{1-\sin\theta}$$
$$= \frac{\cos\theta\cot\theta(1-\sin\theta)}{1-\sin^2\theta}$$
$$= \frac{\cos\theta\frac{\cos\theta}{\sin\theta}(1-\sin\theta)}{\cos^2\theta}$$
$$= \frac{(1-\sin\theta)}{\sin\theta}$$
$$= \frac{1}{\sin\theta} - 1$$
$$= \cos ec\theta - 1$$

Exercise 21 B

Question 1.

Prove that:

(i) $\frac{\cos A}{1-\tan A} + \frac{\sin A}{1-\cot A} = \sin A + \cos A$

(ii) $\frac{\cos^3 A + \sin^3 A}{\cos A + \sin A} + \frac{\cos^3 A - \sin^3 A}{\cos A - \sin A} = 2$

(iii) $\frac{\tan A}{1-\cot A} + \frac{\cot A}{1-\tan A} = \sec A \operatorname{cosec} A + 1$

(iv) $\left(\tan A + \frac{1}{\cos A}\right)^2 + \left(\tan A - \frac{1}{\cos A}\right)^2 = 2\left(\frac{1+\sin^2 A}{1-\sin^2 A}\right)$

(v) $2\sin^2 A + \cos^4 A = 1 + \sin^4 A$

(vi) $\frac{\sin A - \sin B}{\cos A + \cos B} + \frac{\cos A - \cos B}{\sin A + \sin B} = 0$

(vii) $(\operatorname{cosec} A - \sin A)(\sec A - \cos A) = \frac{1}{\tan A + \cot A}$

(viii) $(1 + \tan A \tan B)^2 + (\tan A - \tan B)^2 = \sec^2 A \sec^2 B$

(ix) $\frac{1}{\cos A + \sin A - 1} + \frac{1}{\cos A + \sin A + 1} = \operatorname{cosec} A + \sec A$

Solution:

(i)

$$LHS = \frac{\cos A}{1 - \tan A} + \frac{\sin A}{1 - \cot A}$$

$$= \frac{\cos A}{1 - \frac{\sin A}{\cos A}} + \frac{\sin A}{1 - \frac{\cos A}{\sin A}} = \frac{\cos A}{\frac{\cos A - \sin A}{\cos A}} + \frac{\sin A}{\frac{\sin A - \cos A}{\sin A}}$$

$$= \frac{\cos^2 A}{\cos A - \sin A} + \frac{\sin^2 A}{\sin A - \cos A} = \frac{\cos^2 A - \sin^2 A}{(\cos A - \sin A)}$$

$$= \sin A + \cos A = RHS$$

(ii)

$$\frac{\cos^3 A + \sin^3 A}{\cos A + \sin A} + \frac{\cos^3 A - \sin^3 A}{\cos A - \sin A}$$

$$= \frac{(\cos^3 A + \sin^3 A)(\cos A - \sin A) + (\cos^3 A - \sin^3 A)(\cos A + \sin A)}{\cos^2 A - \sin^2 A}$$

$$= \frac{\cos^4 A - \cos^3 A \sin A + \sin^3 A \cos A - \sin^4 A + \cos^4 A + \cos^3 A \sin A - \sin^3 A \cos A - \sin^4 A}{\cos^2 A - \sin^2 A}$$

$$= \frac{2(\cos^4 A - \sin^4 A)}{\cos^2 A - \sin^2 A}$$

$$= \frac{2(\cos^2 A + \sin^2 A)(\cos^2 A - \sin^2 A)}{(\cos^2 A - \sin^2 A)}$$

$$= 2(\cos^2 A + \sin^2 A)$$

$$= 2 \quad (\because \cos^2 A + \sin^2 A = 1)$$

(iii)

$$\frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A}$$

$$= \frac{\tan A}{1 - \frac{1}{\tan A}} + \frac{\frac{1}{\tan A}}{1 - \tan A}$$

$$= \frac{\tan^2 A}{\tan A - 1} + \frac{1}{\tan A(1 - \tan A)}$$

$$= \frac{\tan^3 A - 1}{\tan A(\tan A - 1)}$$

$$= \frac{(\tan A - 1)(\tan^2 A + 1 + \tan A)}{\tan A(\tan A - 1)}$$

$$= \frac{\sec^2 A + \tan A}{\tan A}$$

$$= \frac{\frac{1}{\cos^2 A}}{\frac{\sin A}{\cos A}} + 1$$

$$= \frac{1}{\sin A \cos A} + 1$$

$$= \sec A \operatorname{cosec} A + 1$$

(iv)

$$\left(\tan A + \frac{1}{\cos A}\right)^2 + \left(\tan A - \frac{1}{\cos A}\right)^2$$

$$= \left(\frac{\sin A + 1}{\cos A}\right)^2 + \left(\frac{\sin A - 1}{\cos A}\right)^2$$

$$= \frac{\sin^2 A + 1 + 2\sin A + \sin^2 A + 1 - 2\sin A}{\cos^2 A}$$

$$= \frac{2 + 2\sin^2 A}{\cos^2 A}$$

$$= 2\left(\frac{1 + \sin^2 A}{1 - \sin^2 A}\right)$$

(v)

$$2\sin^2 A + \cos^4 A$$

$$= 2\sin^2 A + (1 - \sin^2 A)^2$$

$$= 2\sin^2 A + 1 + \sin^4 A - 2\sin^2 A$$

$$= 1 + \sin^4 A$$

(vi)

$$\frac{\sin A - \sin B}{\cos A + \cos B} + \frac{\cos A - \cos B}{\sin A + \sin B}$$

$$= \frac{\sin^2 A - \sin^2 B + \cos^2 A - \cos^2 B}{(\cos A + \cos B)(\sin A + \sin B)}$$

$$= \frac{(\sin^2 A + \cos^2 A) - (\sin^2 B + \cos^2 B)}{(\cos A + \cos B)(\sin A + \sin B)}$$

$$= \frac{1-1}{(\cos A+\cos B)(\sin A+\sin B)}$$

$= 0$

(vii)

LHS

$= (\operatorname{cosec} A - \sin A)(\sec A - \cos A)$

$$= \left(\frac{1}{\sin A} - \sin A\right)\left(\frac{1}{\cos A} - \cos A\right)$$

$$= \left(\frac{1-\sin^2 A}{\sin A}\right)\left(\frac{1-\cos^2 A}{\cos A}\right)$$

$$= \left(\frac{\cos^2 A}{\sin A}\right)\left(\frac{\sin^2 A}{\cos A}\right)$$

$= \sin A \cos A$

$$\text{RHS} = \frac{1}{\tan A + \cot A}$$

$$= \frac{1}{\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}}$$

$$= \frac{\sin A \cos A}{\sin^2 A + \cos^2 A}$$

$= \sin A \cos A$

LHS = RHS

(viii)

$(1+\tan A \tan B)^2 + (\tan A - \tan B)^2$

$= 1 + \tan^2 A \tan^2 B + 2\tan A \tan B + \tan^2 A + \tan^2 B - 2\tan A \tan B$

$= 1 + \tan^2 A + \tan^2 B + \tan^2 A \tan^2 B$

$= \sec^2 A + \tan^2 B(1+\tan^2 A)$

$= \sec^2 A + \tan^2 B \sec^2 A$

$= \sec^2 A(1+\tan^2 B)$

$= \sec^2 A \sec^2 B$

(ix)

$$\frac{1}{(\cos A+\sin A)-1}+\frac{1}{(\cos A+\sin A)+1}$$
$$=\frac{\cos A+\sin A+1+\cos A+\sin A-1}{(\cos A+\sin A)^2-1}$$
$$=\frac{2(\cos A+\sin A)}{\cos^2 A+\sin^2 A+2\cos A\sin A-1}$$
$$=\frac{2(\cos A+\sin A)}{1+2\cos A\sin A-1}=\frac{\cos A+\sin A}{\cos A\sin A}$$
$$=\frac{\cos A}{\cos A\sin A}+\frac{\sin A}{\cos A\sin A}$$
$$=\frac{1}{\sin A}+\frac{1}{\cos A}$$
$$=\operatorname{cosec}A+\sec A$$

Question 2.

If $x\cos A+y\sin A=m$ and $x\sin A-y\cos A=n$, then prove that $x^2+y^2=m^2+n^2$.

Solution:

$$m^2+n^2$$
$$=(x\cos A+y\sin A)^2+(x\sin A-y\cos A)^2$$
$$=x^2\cos^2 A+y^2\sin^2 A+2xy\sin A\cos A$$
$$+x^2\sin^2 A+y^2\cos^2 A-2xy\sin A\cos A$$
$$=x^2(\cos^2 A+\sin^2 A)+y^2(\cos^2 A+\sin^2 A)$$
$$=x^2+y^2$$

Hence, $x^2+y^2=m^2+n^2$.

Question 3.

If $m=a\sec A+b\tan A$ and $n=a\tan A+b\sec A$, prove that $m^2-n^2=a^2-b^2$

Solution:

Given,

$m = a\sec A + b\tan A$ and $n = a\tan A + b\sec A$

$m^2 - n^2 = (a\sec A + b\tan A)^2 - (a\tan A + b\sec A)^2$

$= a^2\sec^2 A + b^2\tan^2 A + 2ab\sec A\tan A$

$-(a^2\tan^2 A + b^2\sec^2 A + 2ab\sec A\tan A)$

$= \sec^2 A(a^2 - b^2) + \tan^2 A(b^2 - a^2)$

$= (a^2 - b^2)[\sec^2 A - \tan^2 A]$

$= (a^2 - b^2)$ [Since $\sec^2 A - \tan^2 A = 1$]

Hence, $m^2 - n^2 = a^2 - b^2$

Question 4.

If $x = r\sin A\cos B$, $y = r\sin A\sin B$ and $z = r\cos A$, prove that $x^2 + y^2 + z^2 = r^2$

Solution:

$LHS = (r\sin A\cos B)^2 + (r\sin A\sin B)^2 + (r\cos A)^2$

$= r^2\sin^2 A\cos^2 B + r^2\sin^2 A\sin^2 B + r^2\cos^2 A$

$= r^2\sin^2 A(\cos^2 B + \sin^2 B) + r^2\cos^2 A$

$= r^2(\sin^2 A + \cos^2 A) = r^2 = RHS$

Question 5.

If $\sin A + \cos A = m$ and $\sec A + \operatorname{cosec} A = n$, prove that $n(m^2-1) = 2m$.

Solution:

Given:

$\sin A + \cos A = m$

and

$\sec A + \operatorname{cosec} A = n$

Consider $L.H.S = n(m^2 - 1)$

$= (\sec A + \operatorname{cosec} A)[(\sin A + \cos A)^2 - 1]$

$= \left(\frac{1}{\cos A} + \frac{1}{\sin A}\right)[\sin^2 A + \cos^2 A + 2\sin A\cos A - 1]$

$= \left(\frac{\cos A + \sin A}{\sin A\cos A}\right)(1 + 2\sin A\cos A - 1)$

$= \frac{(\cos A + \sin A)}{\sin A\cos A}(2\sin A\cos A)$

$= 2(\sin A + \cos A)$

$= 2m = R.H.S.$

Question 6.

If $x = r\cos A\cos B$, $y = r\cos A\sin B$ and $z = r\sin A$, prove that $x^2 + y^2 + z^2 = r^2$

Solution:

$\text{LHS} = (r\cos A\cos B)^2 + (r\cos A\sin B)^2 + (r\sin A)^2$

$= r^2\cos^2 A\cos^2 B + r^2\cos^2 A\sin^2 B + r^2\sin^2 A$

$= r^2\cos^2 A(\cos^2 B + \sin^2 B) + r^2\sin^2 A$

$= r^2(\cos^2 A + \sin^2 A) = r^2 = \text{RHS}$

Question 7.

If $\dfrac{\cos A}{\cos B} = m$ and $\dfrac{\cos A}{\sin B} = n$, show that $(m^2 + n^2)\cos^2 B = n^2$.

Solution:

$\text{LHS} = (m^2 + n^2)\cos^2 B$

$= \left(\dfrac{\cos^2 A}{\cos^2 B} + \dfrac{\cos^2 A}{\sin^2 B}\right)\cos^2 B$

$= \left(\dfrac{\cos^2 A\sin^2 B + \cos^2 A\cos^2 B}{\cos^2 B\sin^2 B}\right)\cos^2 B$

$= \left(\dfrac{\cos^2 A\sin^2 B + \cos^2 A\cos^2 B}{\sin^2 B}\right)$

$= \dfrac{\cos^2 A(\sin^2 B + \cos^2 B)}{\sin^2 B}$

$= \dfrac{\cos^2 A}{\sin^2 B}$

$= n^2$

Hence, $(m^2 + n^2)\cos^2 B = n^2$.

Exercise 21 C

Question 1.

Without using trigonometric tables, show that:

(i) $\tan 10^\circ \tan 15^\circ \tan 75^\circ \tan 80^\circ = 1$

(ii) $\sin 42^\circ \sec 48^\circ + \cos 42^\circ \operatorname{cosec} 48^\circ = 2$

(iii) $\dfrac{\sin 26^\circ}{\sec 64^\circ} + \dfrac{\cos 26^\circ}{\operatorname{cosec} 64^\circ} = 1$

Solution:

(i)

$\tan 10^\circ \tan 15^\circ \tan 75^\circ \tan 80^\circ$

$= \tan(90^\circ - 80^\circ)\tan(90^\circ - 75^\circ)\tan 75^\circ \tan 80^\circ$

$= \cot 80^\circ \cot 75^\circ \tan 75^\circ \tan 80^\circ$

$= 1$ [As $\tan\theta \cot\theta = 1$]

(ii)

$\sin 42^\circ \sec 48^\circ + \cos 42^\circ \operatorname{cosec} 48^\circ = 2$

Consider $\sin 42^\circ \sec 48^\circ + \cos 42^\circ \operatorname{cosec} 48^\circ$

$\Rightarrow \sin 42^\circ \sec(90^\circ - 42^\circ) + \cos 42^\circ \operatorname{cosec}(90^\circ - 42^\circ)$

$\Rightarrow \sin 42^\circ . \operatorname{cosec} 42^\circ + \cos 42^\circ \sec 42^\circ$

$\Rightarrow \sin 42^\circ . \frac{1}{\sin 42^\circ} + \cos 42^\circ \frac{1}{\cos 42^\circ}$

$\Rightarrow 1 + 1 = 2$

(iii)

$\frac{\sin 26^\circ}{\sec 64^\circ} + \frac{\cos 26^\circ}{\operatorname{cosec} 64^\circ}$

$= \frac{\sin 26^\circ}{\sec(90^\circ - 26^\circ)} + \frac{\cos 26^\circ}{\operatorname{cosec}(90^\circ - 26^\circ)}$

$= \frac{\sin 26^\circ}{\operatorname{cosec} 26^\circ} + \frac{\cos 26^\circ}{\sec 26^\circ}$

$= \sin^2 26^\circ + \cos^2 26^\circ$

$= 1$

Question 2.

Express each of the following in terms of angles between 0°and 45°:
(i) sin 59°+ tan 63°
(ii) cosec 68°+ cot 72°
(iii)cos 74°+ sec 67°

Solution:

(i) $\sin 59^\circ + \tan 63^\circ$

$= \sin(90 - 31)^\circ + \tan(90 - 27)^\circ$

$= \cos 31^\circ + \cot 27^\circ$

(ii) $\operatorname{cosec} 68^\circ + \cot 72^\circ$

$= \operatorname{cosec}(90 - 22)^\circ + \cot(90 - 18)^\circ$

$= \sec 22^\circ + \tan 18^\circ$

(iii) $\cos 74^\circ + \sec 67^\circ$

$= \cos(90 - 16)^\circ + \sec(90 - 23)^\circ$

$= \sin 16^\circ + \operatorname{cosec} 23^\circ$

Question 3.

Show that:

(i) $\frac{\sin A}{\sin(90^\circ - A)} + \frac{\cos A}{\cos(90^\circ - A)} = \sec A \operatorname{cosec} A$

(ii) $\sin A \cos A - \frac{\sin A \cos(90^\circ - A) \cos A}{\sec(90^\circ - A)} - \frac{\cos A \sin(90^\circ - A) \sin A}{\operatorname{cosec}(90^\circ - A)} = 0$

Solution:

(i) $\frac{\sin A}{\sin(90^\circ - A)} + \frac{\cos A}{\cos(90^\circ - A)}$

$= \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}$

$= \frac{\sin^2 A + \cos^2 A}{\cos A \sin A}$

$= \frac{1}{\cos A \sin A}$

$= \sec A \operatorname{cosec} A$

(ii) $\sin A \cos A - \frac{\sin A \cos(90^\circ - A) \cos A}{\sec(90^\circ - A)} - \frac{\cos A \sin(90^\circ - A) \sin A}{\operatorname{cosec}(90^\circ - A)}$

$= \sin A \cos A - \frac{\sin A \sin A \cos A}{\operatorname{cosec} A} - \frac{\cos A \cos A \sin A}{\sec A}$

$= \sin A \cos A - \sin^3 A \cos A - \cos^3 A \sin A$

$= \sin A \cos A - \sin A \cos A(\sin^2 A + \cos^2 A)$

$= \sin A \cos A - \sin A \cos A(1)$

$= 0$

Question 4.

For triangle ABC, show that:

(i) $\sin\left(\frac{A+B}{2}\right) = \cos\frac{C}{2}$

(ii) $\tan\left(\frac{B+C}{2}\right) = \cot\frac{A}{2}$

Solution:

(i) We know that for a triangle $\triangle ABC$

$\angle A + \angle B + \angle C = 180^\circ$

$$\frac{\angle B + \angle A}{2} = 90^\circ - \frac{\angle C}{2}$$

$$\sin\left(\frac{A+B}{2}\right) = \sin\left(90^\circ - \frac{C}{2}\right)$$

$$= \cos\left(\frac{C}{2}\right)$$

(ii) We know that for a triangle $\triangle ABC$

$\angle A + \angle B + \angle C = 180^\circ$

$\angle B + \angle C = 180^\circ - \angle A$

$$\frac{\angle B + \angle C}{2} = 90^\circ - \frac{\angle A}{2}$$

$$\tan\left(\frac{B+C}{2}\right) = \tan\left(90^\circ - \frac{A}{2}\right)$$

$$= \cot\left(\frac{A}{2}\right)$$

Question 5.

Evaluate:

(i) $3\frac{\sin 72^\circ}{\cos 18^\circ} - \frac{\sec 32^\circ}{\operatorname{cosec} 58^\circ}$

(ii) $3\cos 80^\circ \operatorname{cosec} 10^\circ + 2\cos 59^\circ \operatorname{cosec} 31^\circ$

(iii) $\frac{\sin 80^\circ}{\cos 10^\circ} + \sin 59^\circ \sec 31^\circ$

(iv) $\tan(55^\circ - A) - \cot(35^\circ + A)$

(v) $\operatorname{cosec}(65^\circ + A) - \sec(25^\circ - A)$

(vi) $2\frac{\tan 57^\circ}{\cot 33^\circ} - \frac{\cot 70^\circ}{\tan 20^\circ} - \sqrt{2}\cos 45^\circ$

(vii) $\frac{\cot^2 41^\circ}{\tan^2 49^\circ} - 2\frac{\sin^2 75^\circ}{\cos^2 15^\circ}$

(viii) $\frac{\cos 70^\circ}{\sin 20^\circ} + \frac{\cos 59^\circ}{\sin 31^\circ} - 8\sin^2 30^\circ$

(ix) $14\sin 30^\circ + 6\cos 60^\circ - 5\tan 45^\circ$

Solution:

(i)

$$3\frac{\sin 72^\circ}{\cos 18^\circ}-\frac{\sec 32^\circ}{\operatorname{cosec} 58^\circ}$$

$$=3\frac{\sin(90^\circ-18^\circ)}{\cos 18^\circ}-\frac{\sec(90^\circ-58^\circ)}{\operatorname{cosec} 58^\circ}$$

$$=3\frac{\cos 18^\circ}{\cos 18^\circ}-\frac{\operatorname{cosec} 58^\circ}{\operatorname{cosec} 58^\circ}=3-1=2$$

(ii) $3\cos 80^\circ \operatorname{cosec} 10^\circ+2\cos 59^\circ \operatorname{cosec} 31^\circ$

$$=3\cos(90^\circ-10^\circ)\operatorname{cosec} 10^\circ+2\cos(90^\circ-31^\circ)\operatorname{cosec} 31^\circ$$

$$=3\sin 10^\circ \operatorname{cosec} 10^\circ+2\sin 31^\circ \operatorname{cosec} 31^\circ$$

$$=3+2=5$$

(iii) $\frac{\sin 80^\circ}{\cos 10^\circ}+\sin 59^\circ \sec 31^\circ$

$$=\frac{\sin(90^\circ-10^\circ)}{\cos 10^\circ}+\sin(90^\circ-31^\circ)\sec 31^\circ$$

$$=\frac{\cos 10^\circ}{\cos 10^\circ}+\frac{\cos 31^\circ}{\cos 31^\circ}$$

$$=1+1=2$$

(iv) $\tan(55^\circ-A)-\cot(35^\circ+A)$

$$=\tan[90^\circ-(35^\circ+A)]-\cot(35^\circ+A)$$

$$=\cot(35^\circ+A)-\cot(35^\circ+A)$$

$$=0$$

(v) $\operatorname{cosec}(65^\circ+A)-\sec(25^\circ-A)$

$$=\operatorname{cosec}[90^\circ-(25^\circ-A)]-\sec(25^\circ-A)$$

$$=\sec(25^\circ-A)-\sec(25^\circ-A)$$

$$=0$$

(vi) $2\frac{\tan 57^\circ}{\cot 33^\circ}-\frac{\cot 70^\circ}{\tan 20^\circ}-\sqrt{2}\cos 45^\circ$

$$=2\frac{\tan(90^\circ-33^\circ)}{\cot 33^\circ}-\frac{\cot(90^\circ-20^\circ)}{\tan 20^\circ}-\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)$$

$$=2\frac{\cot 33^\circ}{\cot 33^\circ}-\frac{\tan 20^\circ}{\tan 20^\circ}-1$$

$$=2-1-1$$

$$=0$$

(vii) $\frac{\cot^2 41^\circ}{\tan^2 49^\circ} - 2\frac{\sin^2 75^\circ}{\cos^2 15^\circ}$

$= \frac{[\cot(90^\circ - 49^\circ)]^2}{\tan^2 49^\circ} - 2\frac{[\sin(90^\circ - 15^\circ)]^2}{\cos^2 15^\circ}$

$= \frac{\tan^2 49^\circ}{\tan^2 49^\circ} - 2\frac{\cos^2 15^\circ}{\cos^2 15^\circ}$

$= 1 - 2 = -1$

(viii) $\frac{\cos 70^\circ}{\sin 20^\circ} + \frac{\cos 59^\circ}{\sin 31^\circ} - 8\sin^2 30^\circ$

$= \frac{\cos(90^\circ - 20^\circ)}{\sin 20^\circ} + \frac{\cos(90^\circ - 31^\circ)}{\sin 31^\circ} - 8\left(\frac{1}{2}\right)^2$

$= \frac{\sin 20^\circ}{\sin 20^\circ} + \frac{\sin 31^\circ}{\sin 31^\circ} - 2$

$= 1 + 1 - 2 = 0$

(ix) $14\sin 30^\circ + 6\cos 60^\circ - 5\tan 45^\circ$

$= 14\left(\frac{1}{2}\right) + 6\left(\frac{1}{2}\right) - 5(1)$

$= 7 + 3 - 5 = 5$

Question 6.

A triangle ABC is right angled at B; find the value of $\frac{\sec A.\operatorname{cosec} C - \tan A.\cot C}{\sin B}$

Solution:

Since, ABC is a right angled triangle, right angled at B.

So, $A + C = 90^\circ$

$\frac{\sec A.\operatorname{cosec} C - \tan A.\cot C}{\sin B}$

$= \frac{\sec(90^\circ - C).\operatorname{cosec} C - \tan(90^\circ - C).\cot C}{\sin 90^\circ}$

$= \frac{\operatorname{cosec} C.\operatorname{cosec} C - \cot C.\cot C}{1}$

$= 1 \qquad [\because \operatorname{cosec}^2\theta - \cot^2\theta = 1]$

Question 7.

Find (in each case, given below) the value of x if:

(i) $\sin x = \sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ$

(ii) $\sin x = \sin 60^\circ \cos 30^\circ + \cos 60^\circ \sin 30^\circ$

(iii) $\cos x = \cos 60^\circ \cos 30^\circ - \sin 60^\circ \sin 30^\circ$

(iv) $\tan x = \dfrac{\tan 60^\circ - \tan 30^\circ}{1 + \tan 60^\circ \tan 30^\circ}$

(v) $\sin 2x = 2 \sin 45^\circ \cos 45^\circ$

(vi) $\sin 3x = 2 \sin 30^\circ \cos 30^\circ$

(vii) $\cos(2x - 6^\circ) = \cos^2 30^\circ - \cos^2 60^\circ$

Solution:

(i) $\sin x = \sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ$

$$\sin x = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$\sin x = \frac{3}{4} - \frac{1}{4} = \frac{1}{2} = \sin 30^\circ$$

Hence, $x = 30^\circ$

(ii) $\sin x = \sin 60^\circ \cos 30^\circ + \cos 60^\circ \sin 30^\circ$

$$\sin x = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$\sin x = \frac{3}{4} + \frac{1}{4} = 1 = \sin 90^\circ$$

Hence, $x = 90^\circ$

(iii) $\cos x = \cos 60^\circ \cos 30^\circ - \sin 60^\circ \sin 30^\circ$

$$\cos x = \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)$$

$$\cos x = 0 = \cos 90^\circ$$

Hence, $x = 90^\circ$

(iv) $\tan x = \dfrac{\tan 60^\circ - \tan 30^\circ}{1 + \tan 60^\circ \tan 30^\circ}$

$$\tan x = \frac{\sqrt{3} - \frac{1}{\sqrt{3}}}{1 + \sqrt{3}.\frac{1}{\sqrt{3}}}$$

(i) $\sin x = \sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ$

$$\sin x = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$\sin x = \frac{3}{4} - \frac{1}{4} = \frac{1}{2} = \sin 30^\circ$$

Hence, $x = 30^\circ$

(ii) $\sin x = \sin 60^\circ \cos 30^\circ + \cos 60^\circ \sin 30^\circ$

$$\sin x = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$\sin x = \frac{3}{4} + \frac{1}{4} = 1 = \sin 90^\circ$$

Hence, $x = 90^\circ$

(iii) $\cos x = \cos 60^\circ \cos 30^\circ - \sin 60^\circ \sin 30^\circ$

$$\cos x = \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)$$

$$\cos x = 0 = \cos 90^\circ$$

Hence, $x = 90^\circ$

(iv) $\tan x = \dfrac{\tan 60^\circ - \tan 30^\circ}{1 + \tan 60^\circ \tan 30^\circ}$

$$\tan x = \frac{\sqrt{3} - \frac{1}{\sqrt{3}}}{1 + \sqrt{3}.\frac{1}{\sqrt{3}}}$$

$$\tan x = \frac{\frac{3-1}{\sqrt{3}}}{1+1} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}} = \tan 30^\circ$$

Hence, $x = 30^\circ$

(v) $\sin 2x = 2\sin 45° \cos 45°$

$$\sin 2x = 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$\sin 2x = 1 = \sin 90°$

$2x = 90°$

Hence, $x = 45°$

(vi) $\sin 3x = 2\sin 30° \cos 30°$

$$\sin 3x = 2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$\sin 3x = \frac{\sqrt{3}}{2} = \sin 60°$$

$3x = 60°$

Hence, $x = 20°$

(vii) $\cos(2x - 6°) = \cos^2 30° - \cos^2 60°$

$\cos(2x - 6) = \cos^2(90° - 60°) - \cos^2 60°$

$\cos(2x - 6) = \sin^2 60° - \cos^2 60°$

$$\cos(2x - 6) = 1 - 2\cos^2 60° = 1 - 2\left(\frac{1}{2}\right)^2 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\cos(2x - 6) = \frac{1}{2}$$

$\cos(2x - 6) = \cos 60°$

$(2x - 6) = 60°$

$2x = 66°$

Hence, $x = 33°$

Question 8.

In each case, given below, find the value of angle A, where $0° \le A \le 90°$.

(i) $\sin(90° - 3A).\operatorname{cosec} 42° = 1$

(ii) $\cos(90° - A).\sec 77° = 1$

Solution:

(i) $\sin(90° - 3A).\text{cosec}42° = 1$

$\cos 3A.\frac{1}{\sin 42°} = 1$

$\cos 3A = \sin 42° = \sin(90° - 48°) = \cos 48°$

$3A = 48°$

$A = 16°$

(ii) $\cos(90° - A).\sec 77° = 1$

$\cos(90° - A).\sec 77° = 1$

$\sin A.\frac{1}{\cos 77°} = 1$

$\sin A = \cos 77° = \cos(90° - 13°) = \sin 13°$

$A = 13°$

Question 9.

Prove that:

(i) $\frac{\cos(90° - \theta)\cos\theta}{\cot\theta} = 1 - \cos^2\theta$

(ii) $\frac{\sin\theta\sin(90° - \theta)}{\cot(90° - \theta)} = 1 - \sin^2\theta$

Solution:

(i)

$$\text{LHS} = \frac{\cos(90° - \theta)\cos\theta}{\cot\theta} = \frac{\sin\theta\cos\theta}{\frac{\cos\theta}{\sin\theta}} = \sin^2\theta = 1 - \cos^2\theta$$

(ii)

$$\text{LHS} = \frac{\sin\theta\sin(90° - \theta)}{\cot(90° - \theta)} = \frac{\sin\theta\cos\theta}{\tan\theta} = \frac{\sin\theta\cos\theta}{\frac{\sin\theta}{\cos\theta}} = \cos^2\theta = 1 - \sin^2\theta$$

Question 10.

Evaluate :

$$\frac{\sin 35° \cos 55° + \cos 35° \sin 55°}{\text{cosec}^2 10° - \tan^2 80°}$$

Solution:

$$\frac{\sin 35^\circ \cos 55^\circ + \cos 35^\circ \sin 55^\circ}{\operatorname{cosec}^2 10^\circ - \tan^2 80^\circ}$$

$$= \frac{\sin 35^\circ . \cos\left(90^\circ - 35^\circ\right) + \cos 35^\circ . \sin\left(90^\circ - 35^\circ\right)}{\operatorname{cosec}^2\left(90^\circ - 80^\circ\right) - \tan^2 80^\circ}$$

$$= \frac{\sin 35^\circ . \sin 35^\circ + \cos 35^\circ . \cos 35^\circ}{\sec^2 80^\circ - \tan^2 80^\circ}$$

$$= \frac{\sin^2 35^\circ + \cos^2 35^\circ}{\sec^2 80^\circ - \tan^2 80^\circ} = \frac{1}{1} = 1$$

Question 11.

Evaluate

$\sin^2 34^\circ + \sin^2 56^\circ + 2\tan 18^\circ \tan 72^\circ - \cot^2 30^\circ$

Solution:

$$\sin^2 34^\circ + \sin^2 56^\circ + 2\tan 18^\circ \tan 72^\circ - \cot^2 30^\circ$$

$$= \sin^2 34^\circ + \sin^2(90^\circ - 34^\circ) + 2\tan 18^\circ \tan(90^\circ - 72^\circ) - \cot^2 30^\circ$$

$$= \sin^2 34^\circ + \cos^2 34^\circ + 2\tan 18^\circ \cot 18^\circ - \cot^2 30^\circ$$

$$= (\sin^2 34^\circ + \cos^2 34^\circ) + 2\tan 18^\circ \times \frac{1}{\tan 18^\circ} - \cot^2 30^\circ$$

$$= 1 + 2 \times 1 - \left(\sqrt{3}\right)^2$$

$$= 1 + 2 - 3$$

$$= 3 - 3$$

$$= 0$$

Question 12.

Without using trigonometrical tables, evaluate:

$\operatorname{cosec}^2 57^\circ - \tan^2 33^\circ + \cos 44^\circ \operatorname{cosec} 46^\circ - \sqrt{2}\cos 45^\circ - \tan^2 60^\circ$

Solution:

$$\operatorname{cosec}^2 57^\circ - \tan^2 33^\circ + \cos 44^\circ \operatorname{cosec} 46^\circ - \sqrt{2}\cos 45^\circ - \tan^2 60^\circ$$
$$= \operatorname{cosec}^2(90^\circ - 33^\circ)^\circ - \tan^2 33^\circ + \cos 44^\circ \operatorname{cosec}(90^\circ - 44^\circ) - \sqrt{2}\cos 45^\circ - \tan^2 60^\circ$$
$$= \sec^2 33^\circ - \tan^2 33^\circ + \cos 44^\circ \sec 44^\circ - \sqrt{2}\cos 45^\circ - \tan^2 60$$
$$= 1 + 1 - \sqrt{2}\cos 45^\circ - \tan^2 60$$
$$= 1 + 1 - \sqrt{2}\left(\frac{1}{\sqrt{2}}\right) - \left(\sqrt{3}\right)^2$$
$$= 2 - 1 - 3$$
$$= -2$$

Exercise 21 D

Question 1.

Use tables to find sine of:

(i) 21°
(ii) 34° 42'
(iii) 47° 32'
(iv) 62° 57'
(v) 10° 20' + 20° 45'

Solution:

(i) sin 21° = 0.3584
(ii) sin 34° 42'= 0.5693
(iii) sin 47° 32'= sin (47° 30' + 2') =0.7373 + 0.0004 = 0.7377
(iv) sin 62° 57' = sin (62° 54' + 3') = 0.8902 + 0.0004 = 0.8906
(v) sin (10° 20' + 20° 45') = sin 30°65' = sin 31°5' = 0.5150 + 0.0012 = 0.5162

Question 2.

Use tables to find cosine of:

(i) 2° 4'
(ii) 8° 12'
(iii) 26° 32'
(iv) 65° 41'
(v) 9° 23' + 15° 54'

Solution:

(i) cos 2° 4' = 0.9994 - 0.0001 = 0.9993
(ii) cos 8° 12' = cos 0.9898
(iii) cos 26° 32' = cos (26° 30' + 2') = 0.8949 - 0.0003 = 0.8946
(iv) cos 65° 41' = cos (65° 36' + 5') = 0.4131 -0.0013 = 0.4118
(v) cos (9° 23' + 15° 54') = cos 24° 77' = cos 25° 17' = cos (25° 12' + 5') = 0.9048 - 0.0006 = 0.9042

Question 3.

Use trigonometrical tables to find tangent of:
(i) 37°
(ii) 42° 18'
(iii) 17° 27'

Solution:

(i) tan 37^{o} = 0.7536
(ii) tan 42^{o} 18' = 0.9099
(iii) tan 17^{o} 27' = tan (17^{o} 24' + 3') = 0.3134 + 0.0010 = 0.3144

Question 4.

Use tables to find the acute angle θ, if the value of sin θ is:
(i) 0.4848
(ii) 0.3827
(iii) 0.6525

Solution:

(i) From the tables, it is clear that sin 29^{o} = 0.4848
Hence, θ = 29^{o}
(ii) From the tables, it is clear that sin 22^{o} 30' = 0.3827
Hence, θ = 22^{o} 30'
(iii) From the tables, it is clear that sin 40^{o} 42' = 0.6521
sin θ - sin 40^{o} 42' = 0.6525 -; 0.6521 = 0.0004
From the tables, diff of 2' = 0.0004
Hence, θ = 40^{o} 42' + 2' = 40^{o} 44'

Question 5.

Use tables to find the acute angle θ, if the value of cos θ is:
(i) 0.9848
(ii) 0.9574
(iii) 0.6885

Solution:

(i) From the tables, it is clear that cos 10° = 0.9848

Hence, θ = 10°

(ii) From the tables, it is clear that cos 16° 48' = 0.9573

cos θ - cos 16° 48' = 0.9574 - 0.9573 = 0.0001

From the tables, diff of 1' = 0.0001

Hence, θ = 16° 48' - 1' = 16° 47'

(iii) From the tables, it is clear that cos 46° 30' = 0.6884

cos q - cos 46° 30' = 0.6885 - 0.6884 = 0.0001

From the tables, diff of 1' = 0.0002

Hence, θ = 46° 30' - 1' = 46° 29'

Question 6.

Use tables to find the acute angle θ, if the value of tan q is:

(i) 0.2419

(ii) 0.4741

(iii) 0.7391

Solution:

(i) From the tables, it is clear that tan 13° 36' = 0.2419

Hence, θ = 13° 36'

(ii) From the tables, it is clear that tan 25° 18' = 0.4727

tan θ - tan 25° 18' = 0.4741 - 0.4727 = 0.0014

From the tables, diff of 4' = 0.0014

Hence, θ = 25° 18' + 4' = 25° 22'

(iii) From the tables, it is clear that tan 36° 24' = 0.7373

tan θ - tan 36° 24' = 0.7391 - 0.7373 = 0.0018

From the tables, diff of 4' = 0.0018

Hence, θ = 36° 24' + 4' = 36° 28'

Exercise 21 E

Question 1.

Prove the following identities:

(i) $\frac{1}{\cos A + \sin A} + \frac{1}{\cos A - \sin A} = \frac{2\cos A}{2\cos^2 A - 1}$

(ii) $\operatorname{cosec} A - \cot A = \frac{\sin A}{1 + \cos A}$

(iii) $1 - \frac{\sin^2 A}{1 + \cos A} = \cos A$

(iv) $\frac{1 - \cos A}{\sin A} + \frac{\sin A}{1 - \cos A} = 2 \operatorname{cosec} A$

(v) $\frac{\cot A}{1-\tan A}+\frac{\tan A}{1-\cot A}=1+\tan A+\cot A$

(vi) $\frac{\cos A}{1+\sin A}+\tan A=\sec A$

(vii) $\frac{\sin A}{1-\cos A}-\cot A=\operatorname{cosec} A$

(viii) $\frac{\sin A-\cos A+1}{\sin A+\cos A-1}=\frac{\cos A}{1-\sin A}$

(ix) $\sqrt{\frac{1+\sin A}{1-\sin A}}=\frac{\cos A}{1-\sin A}$

(x) $\sqrt{\frac{1-\cos A}{1+\cos A}}=\frac{\sin A}{1+\cos A}$

(xi) $\frac{1+(\sec A-\tan A)^2}{\operatorname{cosec} A(\sec A-\tan A)}=2\tan A$

(xii) $\frac{(\operatorname{cosec} A-\cot A)^2+1}{\sec A(\operatorname{cosec} A-\cot A)}=2\cot A$

(xiii) $\cot^2 A\left(\frac{\sec A-1}{1+\sin A}\right)+\sec^2 A\left(\frac{\sin A-1}{1+\sec A}\right)=0$

(xiv) $\frac{(1-2\sin^2 A)^2}{\cos^4 A-\sin^4 A}=2\cos^2 A-1$

(xv) $\sec^4 A(1-\sin^4 A)-2\tan^2 A=1$

(xvi) $\operatorname{cosec}^4 A(1-\cos^4 A)-2\cot^2 A=1$

(xvii) $(1+\tan A+\sec A)(1+\cot A-\operatorname{cosec} A)=2$

Solution:

(i) $\frac{1}{\cos A+\sin A}+\frac{1}{\cos A-\sin A}$

$=\frac{\cos A+\sin A+\cos A-\sin A}{(\cos A+\sin A)(\cos A-\sin A)}$

$=\frac{2\cos A}{\cos^2 A-\sin^2 A}$

$=\frac{2\cos A}{\cos^2 A-(1-\cos^2 A)}$

$=\frac{2\cos A}{2\cos^2 A-1}$

(ii) $\operatorname{cosec} A - \cot A$

$$= \frac{1}{\sin A} - \frac{\cos A}{\sin A}$$

$$= \frac{1 - \cos A}{\sin A}$$

$$= \frac{1 - \cos A}{\sin A} \times \frac{1 + \cos A}{1 + \cos A}$$

$$= \frac{1 - \cos^2 A}{\sin A(1 + \cos A)}$$

$$= \frac{\sin^2 A}{\sin A(1 + \cos A)}$$

$$= \frac{\sin A}{1 + \cos A}$$

(iii) $1 - \frac{\sin^2 A}{1 + \cos A}$

$$= \frac{1 + \cos A - \sin^2 A}{1 + \cos A}$$

$$= \frac{\cos A + \cos^2 A}{1 + \cos A}$$

$$= \frac{\cos A(1 + \cos A)}{1 + \cos A}$$

$$= \cos A$$

(iv) $\frac{1 - \cos A}{\sin A} + \frac{\sin A}{1 - \cos A}$

$$= \frac{(1 - \cos A)^2 + \sin^2 A}{\sin A(1 - \cos A)}$$

$$= \frac{1 + \cos^2 A - 2\cos A + \sin^2 A}{\sin A(1 - \cos A)}$$

$$= \frac{2 - 2\cos A}{\sin A(1 - \cos A)}$$

$$= \frac{2(1 - \cos A)}{\sin A(1 - \cos A)}$$

$$= 2\operatorname{cosec} A$$

(v) $\dfrac{\cot A}{1-\tan A}+\dfrac{\tan A}{1-\cot A}$

$=\dfrac{\dfrac{1}{\tan A}}{1-\tan A}+\dfrac{\tan A}{1-\dfrac{1}{\tan A}}$

$=\dfrac{1}{\tan A(1-\tan A)}+\dfrac{\tan^2 A}{\tan A-1}$

$=\dfrac{1-\tan^3 A}{\tan A(1-\tan A)}$

$=\dfrac{(1-\tan A)(1+\tan A+\tan^2 A)}{\tan A(1-\tan A)}$

$=\dfrac{1+\tan A+\tan^2 A}{\tan A}$

$=\cot A+1+\tan A$

(vi) $\dfrac{\cos A}{1+\sin A}+\tan A$

$=\dfrac{\cos A}{1+\sin A}+\dfrac{\sin A}{\cos A}$

$=\dfrac{\cos^2 A+\sin A+\sin^2 A}{(1+\sin A)\cos A}$

$=\dfrac{1+\sin A}{(1+\sin A)\cos A}$

$=\dfrac{1}{\cos A}$

$=\sec A$

(vii) $\dfrac{\sin A}{1-\cos A}-\cot A$

$=\dfrac{\sin A}{1-\cos A}-\dfrac{\cos A}{\sin A}$

$=\dfrac{\sin^2 A-\cos A+\cos^2 A}{(1-\cos A)\sin A}$

$=\dfrac{1-\cos A}{(1-\cos A)\sin A}$

$= \frac{1}{\sin A}$

$= \operatorname{cosec} A$

(viii) $\frac{\sin A - \cos A + 1}{\sin A + \cos A - 1}$

$= \frac{\sin A - \cos A + 1}{\sin A + \cos A - 1} \times \frac{\sin A - (\cos A - 1)}{\sin A - (\cos A - 1)}$

$= \frac{(\sin A - \cos A + 1)^2}{\sin^2 A - (\cos A - 1)^2}$

$= \frac{\sin^2 A + \cos^2 A + 1 - 2\sin A \cos A - 2\cos A + 2\sin A}{\sin^2 A - \cos^2 A - 1 + 2\cos A}$

$= \frac{1 + 1 - 2\sin A \cos A - 2\cos A + 2\sin A}{-\cos^2 A - \cos^2 A + 2\cos A}$

$= \frac{2(1 - \cos A) + 2\sin A(1 - \cos A)}{2\cos A(1 - \cos A)}$

$= \frac{1 + \sin A}{\cos A}$

$= \frac{1 + \sin A}{\cos A} \times \frac{1 - \sin A}{1 - \sin A}$

$= \frac{\cos^2 A}{\cos A(1 - \sin A)}$

$= \frac{\cos A}{1 - \sin A}$

(ix) $\sqrt{\dfrac{1+\sin A}{1-\sin A}}$

$= \sqrt{\dfrac{1+\sin A}{1-\sin A} \times \dfrac{1-\sin A}{1-\sin A}}$

$= \sqrt{\dfrac{1-\sin^2 A}{(1-\sin A)^2}}$

$= \sqrt{\dfrac{\cos^2 A}{(1-\sin A)^2}}$

$= \dfrac{\cos A}{1-\sin A}$

(x) $\sqrt{\dfrac{1-\cos A}{1+\cos A}}$

$= \sqrt{\dfrac{1-\cos A}{1+\cos A} \times \dfrac{1+\cos A}{1+\cos A}}$

$= \sqrt{\dfrac{1-\cos^2 A}{(1+\cos A)^2}}$

$= \sqrt{\dfrac{\sin^2 A}{(1+\cos A)^2}}$

$= \dfrac{\sin A}{1+\cos A}$

(xi) $\dfrac{1+(\sec A-\tan A)^2}{\operatorname{cosec} A(\sec A-\tan A)}$

$= \dfrac{(\sec^2 A-\tan^2 A)+(\sec A-\tan A)^2}{\operatorname{cosec} A(\sec A-\tan A)}$

$= \dfrac{(\sec A-\tan A)(\sec A+\tan A)+(\sec A-\tan A)^2}{\operatorname{cosec} A(\sec A-\tan A)}$

$= \dfrac{(\sec A+\tan A)+(\sec A-\tan A)}{\operatorname{cosec} A}$

$= \dfrac{2\sec A}{\operatorname{cosec} A}$

$= 2\dfrac{\frac{1}{\cos A}}{\frac{1}{\sin A}}$

$= 2\tan A$

(xii) $\frac{(\operatorname{cosec}A - \cot A)^2 + 1}{\sec A(\operatorname{cosec}A - \cot A)}$

$= \frac{(\operatorname{cosec}A - \cot A)^2 + (\operatorname{cosec}^2A - \cot^2 A)}{\sec A(\operatorname{cosec}A - \cot A)}$

$= \frac{(\operatorname{cosec}A - \cot A)^2 + (\operatorname{cosec}A - \cot A)(\operatorname{cosec}A + \cot A)}{\sec A(\operatorname{cosec}A - \cot A)}$

$= \frac{(\operatorname{cosec}A - \cot A) + (\operatorname{cosec}A + \cot A)}{\sec A}$

$= \frac{2\operatorname{cosec}A}{\sec A}$

$= 2\cot A$

(xiii) $\cot^2 A\left(\frac{\sec A - 1}{1 + \sin A}\right) + \sec^2 A\left(\frac{\sin A - 1}{1 + \sec A}\right)$

$= \cot^2 A\left(\frac{\sec A - 1}{1 + \sin A} \times \frac{\sec A + 1}{\sec A + 1}\right) + \sec^2 A\left(\frac{\sin A - 1}{1 + \sec A}\right)$

$= \cot^2 A\left[\frac{\sec^2 A - 1}{(1 + \sin A)(\sec A + 1)}\right] + \sec^2 A\left(\frac{\sin A - 1}{1 + \sec A}\right)$

$= \cot^2 A\left[\frac{\tan^2 A}{(1 + \sin A)(\sec A + 1)}\right] + \sec^2 A\left(\frac{\sin A - 1}{1 + \sec A}\right)$

$= \frac{1}{(1 + \sin A)(\sec A + 1)} + \sec^2 A\left(\frac{\sin A - 1}{1 + \sec A}\right)$

$= \frac{1 + \sec^2 A(\sin A - 1)(1 + \sin A)}{(1 + \sin A)(\sec A + 1)}$

$= \frac{1 + \sec^2 A(\sin^2 A - 1)}{(1 + \sin A)(\sec A + 1)}$

$= \frac{1 + \sec^2 A(-\cos^2 A)}{(1 + \sin A)(\sec A + 1)}$

$= \frac{1 - 1}{(1 + \sin A)(\sec A + 1)}$

$= 0$

(xiv) $\frac{(1-2\sin^2 A)^2}{\cos^4 A-\sin^4 A}$

$=\frac{(1-2\sin^2 A)^2}{(\cos^2 A-\sin^2 A)(\cos^2 A+\sin^2 A)}$

$=\frac{(1-2\sin^2 A)^2}{1-\sin^2 A-\sin^2 A}$

$=\frac{(1-2\sin^2 A)^2}{1-2\sin^2 A}$

$=1-2\sin^2 A$

$=1-2(1-\cos^2 A)$

$=2\cos^2 A-1$

(xv) $\sec^4 A(1-\sin^4 A)-2\tan^2 A$

$=\sec^4 A(1-\sin^2 A)(1+\sin^2 A)-2\tan^2 A$

$=\sec^4 A(\cos^2 A)(1+\sin^2 A)-2\tan^2 A$

$=\sec^2 A+\frac{\sin^2 A}{\cos^2 A}-2\tan^2 A$

$=\sec^2 A+\tan^2 A-2\tan^2 A$

$=\sec^2 A-\tan^2 A$

$=1$

(xvi) $\operatorname{cosec}^4 A(1-\cos^4 A)-2\cot^2 A$

$=\operatorname{cosec}^4 A(1-\cos^2 A)(1+\cos^2 A)-2\cot^2 A$

$=\operatorname{cosec}^4 A(\sin^2 A)(1+\cos^2 A)-2\cot^2 A$

$=\operatorname{cosec}^2 A(1+\cos^2 A)-2\cot^2 A$

$=\operatorname{cosec}^2 A+\frac{\cos^2 A}{\sin^2 A}-2\cot^2 A$

$=\operatorname{cosec}^2 A+\cot^2 A-2\cot^2 A$

$=\operatorname{cosec}^2 A-\cot^2 A$

$=1$

(xvii) $(1 + \tan A + \sec A)(1 + \cot A - \operatorname{cosec} A)$

$= 1 + \cot A - \operatorname{cosec} A + \tan A + 1 - \sec A + \sec A + \operatorname{cosec} A - \operatorname{cosec} A \sec A$

$= 2 + \frac{\cos A}{\sin A} + \frac{\sin A}{\cos A} - \frac{1}{\sin A \cos A}$

$= 2 + \frac{\cos^2 A + \sin^2 A}{\sin A \cos A} - \frac{1}{\sin A \cos A}$

$= 2 + \frac{1}{\sin A \cos A} - \frac{1}{\sin A \cos A}$

$= 2$

Question 2.

If $\sin A + \cos A = p$ and $\sec A + \operatorname{cosec} A = q$, then prove that:

$q(p^2 - 1) = 2p$

Solution:

$q(p^2 - 1) = (\sec A + \operatorname{cosec} A)[(\sin A + \cos A)^2 - 1]$

$= (\sec A + \operatorname{cosec} A)[(\sin^2 A + \cos^2 A + 2\sin A\cos A) - 1]$

$= (\sec A + \operatorname{cosec} A)[(1 + 2\sin A\cos A) - 1]$

$= (\sec A + \operatorname{cosec} A)(2\sin A\cos A)$

$= 2\sin A + 2\cos A$

$= 2p$

Question 3.

If $x = a\cos\theta$ and $y = b\cot\theta$, show that:

$\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1$

Solution:

$\frac{a^2}{x^2} - \frac{b^2}{y^2}$

$= \frac{a^2}{a^2\cos^2\theta} - \frac{b^2}{b^2\cot^2\theta}$

$= \frac{1}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}$

$= \frac{1-\sin^2\theta}{\cos^2\theta}$

$= \frac{\cos^2\theta}{\cos^2\theta}$

$= 1$

Question 4.

If $\sec A + \tan A = p$, show that:

$\sin A = \frac{p^2-1}{p^2+1}$

Solution:

$\frac{p^2-1}{p^2+1}$

$= \frac{(\sec A + \tan A)^2 - 1}{(\sec A + \tan A)^2 + 1}$

$= \frac{\sec^2 A + \tan^2 A + 2\tan A\sec A - 1}{\sec^2 A + \tan^2 A + 2\tan A\sec A + 1}$

$= \frac{\tan^2 A + \tan^2 A + 2\tan A\sec A}{\sec^2 A + \sec^2 A + 2\tan A\sec A}$

$= \frac{2\tan^2 A + 2\tan A\sec A}{2\sec^2 A + 2\tan A\sec A}$

$= \frac{2\tan A(\tan A + \sec A)}{2\sec A(\tan A + \sec A)}$

$= \sin A$

Question 5.

If tan A = n tan B and sin A = m sin B, prove that:

$\cos^2 A = \frac{m^2-1}{n^2-1}$

Solution:

Given that, tan A = n tan B and sin A = m sin B.

$\Rightarrow n = \dfrac{\tan A}{\tan B}$ and $m = \dfrac{\sin A}{\sin B}$

$\therefore \dfrac{m^2 - 1}{n^2 - 1}$

$= \dfrac{\left(\dfrac{\sin A}{\sin B}\right)^2 - 1}{\left(\dfrac{\tan A}{\tan B}\right)^2 - 1}$

$= \dfrac{\tan^2 B(\sin^2 A - \sin^2 B)}{\sin^2 B(\tan^2 A - \tan^2 B)}$

$= \dfrac{\sin^2 A - \sin^2 B}{\cos^2 B\left(\dfrac{\sin^2 A}{\cos^2 A} - \dfrac{\sin^2 B}{\cos^2 B}\right)}$

$= \dfrac{\cos^2 A(\sin^2 A - \sin^2 B)}{\sin^2 A\cos^2 B - (1 - \cos^2 B)\cos^2 A}$

$= \dfrac{\cos^2 A(1 - \cos^2 A - 1 + \cos^2 B)}{\cos^2 B(\sin^2 A + \cos^2 A) - \cos^2 A}$

$= \dfrac{\cos^2 A(\cos^2 B - \cos^2 A)}{\cos^2 B - \cos^2 A}$

$= \cos^2 A$

Question 6.

(i) If 2 sinA - 1 = 0, show that:
$\sin 3A = 3\sin A - 4\sin^3 A$
(ii) If $4\cos^2 A - 3 = 0$, show that:
$\cos 3A = 4\cos^3 A - 3\cos A$

Solution:

(i) $2\sin A - 1 = 0$

$\Rightarrow \sin A = \frac{1}{2}$

We know $\sin 30^\circ = \frac{1}{2}$

So, $A = 30^\circ$

$\text{LHS} = \sin 3A = \sin 90^\circ = 1$

$\text{RHS} = 3\sin A - 4\sin^3 A$

$= 3\sin 30^\circ - 4\sin^3 30^\circ$

$= 3\left(\frac{1}{2}\right) - 4\left(\frac{1}{2}\right)^3$

$= \frac{3}{2} - \frac{1}{2} = 1$

LHS = RHS

(ii)

$4\cos^2 A - 3 = 0$

$\Rightarrow 4\cos^2 A = 3$

$\Rightarrow \cos^2 A = \frac{3}{4}$

$\Rightarrow \cos A = \frac{\sqrt{3}}{2}$

We know $\cos 30^\circ = \frac{\sqrt{3}}{2}$

So, $A = 30^\circ$

$\text{LHS} = \cos 3A = \cos 90^\circ = 0$

$\text{RHS} = 4\cos^3 A - 3\cos A$

$= 4\cos^3 30^\circ - 3\cos 30^\circ$

$= 4\left(\frac{\sqrt{3}}{2}\right)^3 - 3\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{3\sqrt{3}}{2} - \frac{3\sqrt{3}}{2} = 0$

LHS = RHS

Question 7.

Evaluate:

(i) $2\left(\frac{\tan 35^\circ}{\cot 55^\circ}\right)^2 + \left(\frac{\cot 55^\circ}{\tan 35^\circ}\right)^2 - 3\left(\frac{\sec 40^\circ}{\operatorname{cosec} 50^\circ}\right)$

(ii) $\sec 26^\circ \sin 64^\circ + \frac{\text{coe}\sec 33^\circ}{\sec 57^\circ}$

(iii) $\frac{5\sin 66^\circ}{\cos 24^\circ} - \frac{2\cot 85^\circ}{\tan 5^\circ}$

(iv) $\cos 40^\circ \operatorname{cosec} 50^\circ + \sin 50^\circ \sec 40^\circ$

(v) $\sin 27^\circ \sin 63^\circ - \cos 63^\circ \cos 27^\circ$

(vi) $\frac{3\sin 72^\circ}{\cos 18^\circ} - \frac{\sec 32^\circ}{\operatorname{cosec} 58^\circ}$

(vii) $3\cos 80^\circ \operatorname{cosec} 10^\circ + 2\cos 59^\circ \operatorname{cosec} 31^\circ$

(viii) $\frac{\cos 75^\circ}{\sin 15^\circ} + \frac{\sin 12^\circ}{\cos 78^\circ} - \frac{\cos 18^\circ}{\sin 72^\circ}$

Solution:

(i) $2\left(\frac{\tan 35^\circ}{\cot 55^\circ}\right)^2 + \left(\frac{\cot 55^\circ}{\tan 35^\circ}\right)^2 - 3\left(\frac{\sec 40^\circ}{\operatorname{cosec} 50^\circ}\right)$

$= 2\left(\frac{\tan(90^\circ - 55^\circ)}{\cot 55^\circ}\right)^2 + \left(\frac{\cot(90^\circ - 35^\circ)}{\tan 35^\circ}\right)^2 - 3\left(\frac{\sec(90^\circ - 50^\circ)}{\operatorname{cosec} 50^\circ}\right)$

$= 2\left(\frac{\cot 55^\circ}{\cot 55^\circ}\right)^2 + \left(\frac{\tan 35^\circ}{\tan 35^\circ}\right)^2 - 3\left(\frac{\operatorname{cosec} 50^\circ}{\operatorname{cosec} 50^\circ}\right)$

$= 2(1)^2 + 1^2 + -3$

$= 2 + 1 - 3$

$= 0$

(ii) $\sec 26^\circ \sin 64^\circ + \frac{\text{coe}\sec 33^\circ}{\sec 57^\circ}$

$= \sec(90^\circ - 64^\circ)\sin 64^\circ + \frac{\text{coe}\sec(90^\circ - 57^\circ)}{\sec 57^\circ}$

$= \operatorname{cosec} 64^\circ \sin 64^\circ + \frac{\sec 57^\circ}{\sec 57^\circ}$

$= 1 + 1 = 2$

(iii) $\frac{5\sin 66^\circ}{\cos 24^\circ} - \frac{2\cot 85^\circ}{\tan 5^\circ}$

$= \frac{5\sin(90^\circ - 24^\circ)}{\cos 24^\circ} - \frac{2\cot(90^\circ - 5^\circ)}{\tan 5^\circ}$

$= \frac{5\cos 24^\circ}{\cos 24^\circ} - \frac{2\tan 5^\circ}{\tan 5^\circ}$

$= 5 - 2 = 3$

(iv) $\cos 40^\circ \operatorname{cosec} 50^\circ + \sin 50^\circ \sec 40^\circ$

$= \cos(90^\circ - 50^\circ)\operatorname{cosec} 50^\circ + \sin(90^\circ - 40^\circ)\sec 40^\circ$

$= \sin 50^\circ \operatorname{cosec} 50^\circ + \cos 40^\circ \sec 40^\circ$

$= 1 + 1 = 2$

(v) $\sin 27^\circ \sin 63^\circ - \cos 63^\circ \cos 27^\circ$

$= \sin(90^\circ - 63^\circ)\sin 63^\circ - \cos 63^\circ \cos(90^\circ - 63^\circ)$

$= \cos 63^\circ \sin 63^\circ - \cos 63^\circ \sin 63^\circ$

$= 0$

(vi) $\frac{3\sin 72^\circ}{\cos 18^\circ} - \frac{\sec 32^\circ}{\operatorname{cosec} 58^\circ}$

$= \frac{3\sin(90^\circ - 18^\circ)}{\cos 18^\circ} - \frac{\sec(90^\circ - 58^\circ)}{\operatorname{cosec} 58^\circ}$

$= \frac{3\cos 18^\circ}{\cos 18^\circ} - \frac{\operatorname{cosec} 58^\circ}{\operatorname{cosec} 58^\circ}$

$= 3 - 1 = 2$

(vii) $3\cos 80^\circ \operatorname{cosec} 10^\circ + 2\cos 59^\circ \operatorname{cosec} 31^\circ$

$= 3\cos(90^\circ - 10^\circ)\operatorname{cosec} 10^\circ + 2\cos(90^\circ - 31^\circ)\operatorname{cosec} 31^\circ$

$= 3\sin 10^\circ \operatorname{cosec} 10^\circ + 2\sin 31^\circ \operatorname{cosec} 31^\circ$

$= 3 + 2 = 5$

(viii) $\frac{\cos 75°}{\sin 15°} + \frac{\sin 12°}{\cos 78°} - \frac{\cos 18°}{\sin 72°}$

$= \frac{\cos(90° - 15°)}{\sin 15°} + \frac{\sin(90° - 78°)}{\cos 78°} - \frac{\cos(90° - 72°)}{\sin 72°}$

$= \frac{\sin 15°}{\sin 15°} + \frac{\cos 78°}{\cos 78°} - \frac{\sin 72°}{\sin 72°}$

$= 1 + 1 - 1 = 1$

Question 8.

Prove that:

(i) $\tan(55° + x) = \cot(35° - x)$

(ii) $\sec(70° - \theta) = \operatorname{cosec}(20° + \theta)$

(iii) $\sin(28° + A) = \cos(62° - A)$

(iv) $\frac{1}{1 + \cos(90° - A)} + \frac{1}{1 - \cos(90° - A)} = 2\operatorname{cosec}^2(90° - A)$

(v) $\frac{1}{1 + \sin(90° - A)} + \frac{1}{1 - \sin(90° - A)} = 2\sec^2(90° - A)$

Solution:

(i) $\tan(55° + x) = \tan[90° - (35° - x)] = \cot(35° - x)$

(ii) $\sec(70° - \theta) = \sec[90° - (20° + \theta)] = \operatorname{cosec}(20° + \theta)$

(iii) $\sin(28° + A) = \sin[90° - (62° - A)] = \cos(62° - A)$

(iv) $\frac{1}{1 + \cos(90° - A)} + \frac{1}{1 - \cos(90° - A)}$

$= \frac{1}{1 + \sin A} + \frac{1}{1 - \sin A}$

$= \frac{1 - \sin A + 1 + \sin A}{(1 + \sin A)(1 - \sin A)}$

$= \frac{2}{1 - \sin^2 A}$

$= \frac{2}{\cos^2 A}$

$= 2\sec^2 A$

$= 2\operatorname{cosec}^2(90° - A)$

(v) $\frac{1}{1+\sin(90^\circ - A)} + \frac{1}{1-\sin(90^\circ - A)}$

$= \frac{1}{1+\cos A} + \frac{1}{1-\cos A}$

$= \frac{1-\cos A + 1 + \cos A}{(1+\cos A)(1-\cos A)}$

$= \frac{2}{1-\cos^2 A}$

$= 2\text{cosec}^2 A$

$= 2\sec^2(90^\circ - A)$

Question 9.

If A and B are complementary angles, prove that:

(i) $\cot B + \cos B = \sec A \cos B(1 + \sin B)$

(ii) $\cot A \cot B - \sin A \cos B - \cos A \sin B = 0$

(iii) $\text{cosec}^2 A + \text{cosec}^2 B = \text{cosec}^2 A \,\text{cosec}^2 B$

(iv) $\frac{\sin A + \sin B}{\sin A - \sin B} + \frac{\cos B - \cos A}{\cos B + \cos A} = \frac{2}{2\sin^2 A - 1}$

Solution:

Since, A and B are complementary angles, $A + B = 90^\circ$

(i)

$\cot B + \cos B$

$= \cot(90^\circ - A) + \cos(90^\circ - A)$

$= \tan A + \sin A$

$= \frac{\sin A}{\cos A} + \sin A$

$= \frac{\sin A + \sin A \cos A}{\cos A}$

$= \frac{\sin A(1 + \cos A)}{\cos A}$

$= \sec A \sin A(1 + \cos A)$

$= \sec A \sin(90^\circ - B)[1 + \cos(90^\circ - B)]$

$= \sec A \cos B(1 + \sin B)$

(ii)

$\cot A \cot B - \sin A \cos B - \cos A \sin B$

$= \cot A \cot(90 - A) - \sin A \cos(90 - A) - \cos A \sin(90 - A)$

$= \cot A \tan A - \sin A \sin A - \cos A \cos A$

$= 1 - (\sin^2 A + \cos^2 A)$

$= 1 - 1$

$= 0$

(iii)

$\text{cosec}^2 A + \text{cosec}^2 B$

$= \text{cosec}^2 A + [\text{cosec}(90 - A)]^2$

$= \text{cosec}^2 A + \sec^2 A$

$= \dfrac{1}{\sin^2 A} + \dfrac{1}{\cos^2 A}$

$= \dfrac{\cos^2 A + \sin^2 A}{\sin^2 A \cos^2 A}$

$= \dfrac{1}{\sin^2 A \cos^2 A}$

= $\text{cosec}^2 A\,[\sec(90$Âº $- B)]^2$

$= \text{cosec}^2 A\, \text{cosec}^2 B$

(iv) $\dfrac{\sin A + \sin B}{\sin A - \sin B} + \dfrac{\cos B - \cos A}{\cos B + \cos A}$

$= \dfrac{\sin A + \sin B}{\sin A - \sin B} + \dfrac{\cos(90^\circ - A) - \cos(90^\circ - B)}{\cos(90^\circ - A) + \cos(90^\circ - B)}$

$= \dfrac{\sin A + \sin B}{\sin A - \sin B} + \dfrac{\sin A - \sin B}{\sin A + \sin B}$

$= \dfrac{(\sin A + \sin B)^2 + (\sin A - \sin B)^2}{(\sin A - \sin B)(\sin A + \sin B)}$

$= \dfrac{\sin^2 A + \sin^2 B + 2\sin A \sin B + \sin^2 A + \sin^2 B - 2\sin A \sin B}{\sin^2 A - \sin^2 B}$

$$= 2\frac{\sin^2 A + \sin^2 B}{\sin^2 A - \sin^2 B}$$

$$= 2\frac{\sin^2 A + \sin^2(90^\circ - A)}{\sin^2 A - \sin^2(90^\circ - A)}$$

$$= 2\frac{\sin^2 A + \cos^2 A}{\sin^2 A - \cos^2 A}$$

$$= \frac{2}{\sin^2 A - (1 - \sin^2 A)}$$

$$= \frac{2}{2\sin^2 A - 1}$$

(iv) $\frac{\sin A + \sin B}{\sin A - \sin B} + \frac{\cos B - \cos A}{\cos B + \cos A}$

$$= \frac{\sin A + \sin B}{\sin A - \sin B} + \frac{\cos(90^\circ - A) - \cos(90^\circ - B)}{\cos(90^\circ - A) + \cos(90^\circ - B)}$$

$$= \frac{\sin A + \sin B}{\sin A - \sin B} + \frac{\sin A - \sin B}{\sin A + \sin B}$$

$$= \frac{(\sin A + \sin B)^2 + (\sin A - \sin B)^2}{(\sin A - \sin B)(\sin A + \sin B)}$$

$$= \frac{\sin^2 A + \sin^2 B + 2\sin A \sin B + \sin^2 A + \sin^2 B - 2\sin A \sin B}{\sin^2 A - \sin^2 B}$$

$$= 2\frac{\sin^2 A + \sin^2 B}{\sin^2 A - \sin^2 B}$$

$$= 2\frac{\sin^2 A + \sin^2(90^\circ - A)}{\sin^2 A - \sin^2(90^\circ - A)}$$

$$= 2\frac{\sin^2 A + \cos^2 A}{\sin^2 A - \cos^2 A}$$

$$= \frac{2}{\sin^2 A - (1 - \sin^2 A)}$$

$$= \frac{2}{2\sin^2 A - 1}$$

Question 10.

$\frac{\cot A - 1}{2 - \sec^2 A} = \frac{\cot A}{1 + \tan A}$

Solution:

To prove that : $\frac{\cot A - 1}{2 - \sec^2 A} = \frac{\cot A}{1 + \tan A}$

$$\text{L.H.S} = \frac{\cot A - 1}{2 - \sec^2 A}$$

$$= \frac{\frac{1}{\tan A} - 1}{2 - (1 + \tan^2 A)}$$

$$= \frac{1 - \tan A}{\tan A\,(1 - \tan^2 A)}$$

$$= \frac{(1 - \tan A)}{\tan A\,(1 - \tan A)(1 + \tan A)}$$

$$= \frac{1}{\tan A(1 + \tan A)}$$

$$= \frac{1}{\tan A} \times \frac{1}{(1 + \tan A)}$$

$$= \frac{\cot A}{1 + \tan A}$$

$$= \text{R.H.S}$$

Hence proved.

Question 11.

Prove that:

(i) $\frac{1}{\sin A - \cos A} - \frac{1}{\sin A + \cos A} = \frac{2\cos A}{2\sin^2 A - 1}$

(ii) $\frac{\cot^2 A}{\operatorname{cosec} A - 1} - 1 = \operatorname{cosec} A$

(iii) $\frac{\cos A}{1 + \sin A} = \sec A - \tan A$

(iv) $\cos A(1 + \cot A) + \sin A(1 + \tan A) = \sec A + \operatorname{cosec} A$

(v) $(\sin A - \cos A)(1 + \tan A + \cot A) = \frac{\sec A}{\operatorname{cosec}^2 A} - \frac{\operatorname{cosec} A}{\sec^2 A}$

(vi) $\sqrt{\sec^2 A + \operatorname{cosec}^2 A} = \tan A + \cot A$

(vii) $(\sin A + \cos A)(\sec A + \operatorname{cosec} A) = 2 + \sec A \operatorname{cosec} A$

(viii) $(\tan A + \cot A)(\operatorname{cosec} A - \sin A)(\sec A - \cos A) = 1$

(ix) $\cot^2 A - \cot^2 B = \dfrac{\cos^2 A - \cos^2 B}{\sin^2 A \sin^2 B} = \operatorname{cosec}^2 A - \operatorname{cosec}^2 B$

Solution:

(i) $\dfrac{1}{\sin A - \cos A} - \dfrac{1}{\sin A + \cos A}$

$= \dfrac{\sin A + \cos A - \sin A + \cos A}{(\sin A - \cos A)(\sin A + \cos A)}$

$= \dfrac{2\cos A}{\sin^2 A - \cos^2 A}$

$= \dfrac{2\cos A}{\sin^2 A - (1 - \sin^2 A)}$

$= \dfrac{2\cos A}{2\sin^2 A - 1}$

(ii) $\dfrac{\cot^2 A}{\operatorname{cosec} A - 1} - 1$

$= \dfrac{\cot^2 A - \operatorname{cosec} A + 1}{\operatorname{cosec} A - 1}$

$= \dfrac{-\operatorname{cosec} A + \operatorname{cosec}^2 A}{\operatorname{cosec} A - 1}$

$= \dfrac{\operatorname{cosec} A(\operatorname{cosec} A - 1)}{\operatorname{cosec} A - 1}$

$= \operatorname{cosec} A$

(iii) $\frac{\cos A}{1+\sin A}$

$= \frac{\cos A}{1+\sin A} \times \frac{1-\sin A}{1-\sin A}$

$= \frac{\cos A(1-\sin A)}{1-\sin^2 A}$

$= \frac{\cos A(1-\sin A)}{\cos^2 A}$

$= \frac{1-\sin A}{\cos A}$

$= \sec A - \tan A$

(iv) $\cos A(1+\cot A)+\sin A(1+\tan A)$

$= \cos A + \frac{\cos^2 A}{\sin A} + \sin A + \frac{\sin^2 A}{\cos A}$

$= \sin A + \frac{\cos^2 A}{\sin A} + \cos A + \frac{\sin^2 A}{\cos A}$

$= \left(\frac{\cos^2 A+\sin^2 A}{\sin A}\right) + \left(\frac{\cos^2 A+\sin^2 A}{\cos A}\right)$

$= \frac{1}{\sin A} + \frac{1}{\cos A}$

$= \operatorname{cosec} A + \sec A$

(v) $(\sin A - \cos A)(1+\tan A+\cot A)$

$= \sin A + \frac{\sin^2 A}{\cos A} + \cos A - \cos A - \sin A - \frac{\cos^2 A}{\sin A}$

$= \frac{\sin^2 A}{\cos A} - \frac{\cos^2 A}{\sin A}$

$= \frac{\sec A}{\operatorname{cosec}^2 A} - \frac{\operatorname{cosec} A}{\sec^2 A}$

(vi) $LHS = \sqrt{\sec^2 A + \operatorname{cosec}^2 A}$

$= \sqrt{\frac{1}{\cos^2 A} + \frac{1}{\sin^2 A}}$

$= \sqrt{\frac{\sin^2 A + \cos^2 A}{\sin^2 A \cos^2 A}}$

$= \sqrt{\frac{1}{\sin^2 A \cos^2 A}}$

$= \frac{1}{\sin A \cos A}$

$RHS = \tan A + \cot A$

$= \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}$

$= \frac{\sin^2 A + \cos^2 A}{\sin A \cos A}$

$= \frac{1}{\sin A \cos A}$

$LHS = RHS$

(vii) $(\sin A + \cos A)(\sec A + \operatorname{cosec} A)$

$= \frac{\sin A}{\cos A} + 1 + 1 + \frac{\cos A}{\sin A}$

$= 2 + \frac{\cos^2 A + \sin^2 A}{\sin A \cos A}$

$= 2 + \frac{1}{\sin A \cos A}$

$= 2 + \sec A \operatorname{cosec} A$

(viii) $(\tan A + \cot A)(\operatorname{cosec} A - \sin A)(\sec A - \cos A)$

$$= \left(\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}\right)\left(\frac{1}{\sin A} - \sin A\right)\left(\frac{1}{\cos A} - \cos A\right)$$

$$= \left(\frac{\sin^2 A + \cos^2 A}{\sin A \cos A}\right)\left(\frac{1 - \sin^2 A}{\sin A}\right)\left(\frac{1 - \cos^2 A}{\cos A}\right)$$

$$= \left(\frac{1}{\sin A \cos A}\right)\left(\frac{\cos^2 A}{\sin A}\right)\left(\frac{\sin^2 A}{\cos A}\right)$$

$$= 1$$

(ix) $\cot^2 A - \cot^2 B$

$$= \frac{\cos^2 A}{\sin^2 A} - \frac{\cos^2 B}{\sin^2 B}$$

$$= \frac{\cos^2 A \sin^2 B - \cos^2 B \sin^2 A}{\sin^2 A \sin^2 B}$$

$$= \frac{\cos^2 A(1 - \cos^2 B) - \cos^2 B(1 - \cos^2 A)}{\sin^2 A \sin^2 B}$$

$$= \frac{\cos^2 A - \cos^2 A \cos^2 B - \cos^2 B + \cos^2 B \cos^2 A}{\sin^2 A \sin^2 B}$$

$$= \frac{\cos^2 A - \cos^2 B}{\sin^2 A \sin^2 B}$$

$$= \frac{1 - \sin^2 A - 1 + \sin^2 B}{\sin^2 A \sin^2 B}$$

$$= \frac{-\sin^2 A + \sin^2 B}{\sin^2 A \sin^2 B}$$

$$= \frac{\sin^2 B}{\sin^2 A \sin^2 B} - \frac{\sin^2 A}{\sin^2 A \sin^2 B}$$

$$= \frac{1}{\sin^2 A} - \frac{1}{\sin^2 B}$$

$$= \operatorname{cosec}^2 A - \operatorname{cosec}^2 B$$

Question 12.

If $4\cos^2 A - 3 = 0$ and $0° \le A \le 90°$, then prove that:

(i) $\sin 3A = 3 \sin A - 4 \sin^3 A$

(ii) $\cos 3A = 4 \cos^3 A - 3 \cos A$

Solution:

$4\cos^2 A - 3 = 0$

$\cos A = \frac{\sqrt{3}}{2}$

We know $\cos 30° = \frac{\sqrt{3}}{2}$

So, $A = 30°$

(i)

$LHS = \sin 3A = \sin 90° = 1$

$RHS = 3\sin A - 4\sin^3 A$

$= 3\sin 30° - 4\sin^3 30°$

$= 3 \times \frac{1}{2} - 4 \times \left(\frac{1}{2}\right)^3$

$= \frac{3}{2} - \frac{1}{2}$

$= 1$

LHS = RHS

(ii)

$LHS = \cos 3A = \cos 90° = 0$

$RHS = 4\cos^3 A - 3\cos A$

$= 4\cos^3 30° - 3\cos 30°$

$= 4\left(\frac{\sqrt{3}}{2}\right)^3 - 3\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{3\sqrt{3}}{2} - \frac{3\sqrt{3}}{2} = 0$

LHS = RHS

Question 13.

Find A, if $0° \leq A \leq 90°$ and:

(i) $2\cos^2 A - 1 = 0$

(ii) $\sin 3A - 1 = 0$

(iii) $4\sin^2 A - 3 = 0$

(iv) $\cos^2 A - \cos A = 0$

(v) $2\cos^2 A + \cos A - 1 = 0$

Solution:

(i) $2\cos^2 A - 1 = 0$

$\Rightarrow \cos^2 A = \frac{1}{2}$

$\Rightarrow \cos A = \frac{1}{\sqrt{2}}$

We know $\cos 45^\circ = \frac{1}{\sqrt{2}}$

Hence, $A = 45^\circ$

(ii) $\sin 3A - 1 = 0$

$\Rightarrow \sin 3A = 1$

We know $\sin 90^\circ = 1$

$\therefore 3A = 90^\circ$

Hence, $A = 30^\circ$

(iii) $4\sin^2 A - 3 = 0$

$\Rightarrow \sin^2 A = \frac{3}{4}$

$\Rightarrow \sin A = \frac{\sqrt{3}}{2}$

We know $\sin 60^\circ = \frac{\sqrt{3}}{2}$

Hence, $A = 60^\circ$

(iv) $\cos^2 A - \cos A = 0$

$\Rightarrow \cos A(\cos A - 1) = 0$

$\Rightarrow \cos A = 0$ or $\cos A = 1$

We know $\cos 90^\circ = 0$ and $\cos 0^\circ = 1$

Hence, $A = 90^\circ$ or 0°

(v) $2\cos^2 A + \cos A - 1 = 0$

$\Rightarrow 2\cos^2 A + 2\cos A - \cos A - 1 = 0$

$\Rightarrow 2\cos A(\cos A + 1) - 1(\cos A + 1) = 0$

$\Rightarrow (2\cos A - 1)(\cos A + 1) = 0$

$\Rightarrow \cos A = \frac{1}{2}$ or $\cos A = -1$

We know $\cos 60° = \frac{1}{2}$

We also know that for no value of $A(0° \le A \le 90°)$, $\cos A = -1$.

Hence, $A = 60°$

Question 14.

If $0° < A < 90°$; find A, if:

(i) $\frac{\cos A}{1 - \sin A} + \frac{\cos A}{1 + \sin A} = 4$

(ii) $\frac{\sin A}{\sec A - 1} + \frac{\sin A}{\sec A + 1} = 2$

Solution:

(i) $\frac{\cos A}{1 - \sin A} + \frac{\cos A}{1 + \sin A} = 4$

$\Rightarrow \frac{\cos A + \cos A \sin A + \cos A - \sin A \cos A}{(1 - \sin A)(1 + \sin A)} = 4$

$\Rightarrow \frac{2\cos A}{1 - \sin^2 A} = 4$

$\Rightarrow \frac{2\cos A}{\cos^2 A} = 4$

$\Rightarrow \frac{1}{\cos A} = 2$

$\Rightarrow \cos A = \frac{1}{2}$

We know $\cos 60° = \frac{1}{2}$

Hence, $A = 60°$

(ii) $\dfrac{\sin A}{\sec A - 1} + \dfrac{\sin A}{\sec A + 1} = 2$

$\Rightarrow \dfrac{\sin A \sec A + \sin A + \sec A \sin A - \sin A}{(\sec A - 1)(\sec A + 1)} = 2$

$\Rightarrow \dfrac{2\sin A \sec A}{\sec^2 A - 1} = 2$

$\Rightarrow \dfrac{\sin A \sec A}{\tan^2 A} = 1$

$\Rightarrow \dfrac{\cos A}{\sin A} = 1$

$\Rightarrow \cot A = 1$

We know $\cot 45^\circ = 1$

Hence, $A = 45^\circ$

(i) $\dfrac{\cos A}{1 - \sin A} + \dfrac{\cos A}{1 + \sin A} = 4$

$\Rightarrow \dfrac{\cos A + \cos A \sin A + \cos A - \sin A \cos A}{(1 - \sin A)(1 + \sin A)} = 4$

$\Rightarrow \dfrac{2\cos A}{1 - \sin^2 A} = 4$

$\Rightarrow \dfrac{2\cos A}{\cos^2 A} = 4$

$\Rightarrow \dfrac{1}{\cos A} = 2$

$\Rightarrow \cos A = \dfrac{1}{2}$

We know $\cos 60^\circ = \dfrac{1}{2}$

Hence, $A = 60^\circ$

(ii) $\dfrac{\sin A}{\sec A - 1} + \dfrac{\sin A}{\sec A + 1} = 2$

$\Rightarrow \dfrac{\sin A \sec A + \sin A + \sec A \sin A - \sin A}{(\sec A - 1)(\sec A + 1)} = 2$

$\Rightarrow \dfrac{2\sin A \sec A}{\sec^2 A - 1} = 2$

$$\Rightarrow \frac{\sin A \sec A}{\tan^2 A} = 1$$

$$\Rightarrow \frac{\cos A}{\sin A} = 1$$

$$\Rightarrow \cot A = 1$$

We know $\cot 45^\circ = 1$

Hence, $A = 45^\circ$

Question 15.

Prove that:
$(\operatorname{cosec} A - \sin A)(\sec A - \cos A)\sec^2 A = \tan A$

Solution:

L.H.S.,

$$(\operatorname{cosec} A - \sin A)(\sec A - \cos A)\sec^2 A$$

$$= \left(\frac{1}{\sin A} - \sin A\right)\left(\frac{1}{\cos A} - \cos A\right)\sec^2 A$$

$$= \left(\frac{1 - \sin^2 A}{\sin A}\right)\left(\frac{1 - \cos^2 A}{\cos A}\right)\sec^2 A$$

$$= \left(\frac{\cos^2 A}{\sin A}\right)\left(\frac{\sin^2 A}{\cos A}\right)\sec^2 A$$

$$= \frac{\sin A}{\cos A} = \tan A = \text{R.H.S.}$$

Question 16.

Prove the identity $(\sin\theta + \cos\theta)(\tan\theta + \cot\theta) = \sec\theta + \operatorname{cosec}\theta$.

Solution:

$$\begin{aligned} L.H.S. &= (\sin\theta+\cos\theta)(\tan\theta+\cot\theta) \\ &= (\sin\theta+\cos\theta)\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right) \\ &= (\sin\theta+\cos\theta)\left(\frac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta}\right) \\ &= \frac{\sin\theta+\cos\theta}{\cos\theta\sin\theta} \\ &= \frac{\sin\theta}{\cos\theta\sin\theta}+\frac{\cos\theta}{\cos\theta\sin\theta} \\ &= \frac{1}{\cos\theta}+\frac{1}{\sin\theta} \\ &= \sec\theta+\operatorname{cosec}\theta \\ &= R.H.S. \end{aligned}$$

Question 17.
Evaluate without using trigonometric tables,
$\sin^2 28° + \sin^2 62° + \tan^2 38° - \cot^2 52° + \frac{1}{4}\sec^2 30°$

Solution:

$$\sin^2 28° + \sin^2 62° + \tan^2 38° - \cot^2 52° + \frac{1}{4}\sec^2 30°$$

$$= \sin^2 28° + [\sin(90-28)°]^2 + \tan^2 38° - [\cot(90-38)°]^2 + \frac{1}{4}\sec^2 30°$$

$$= \sin^2 28° + \cos^2 28° + \tan^2 38° - \tan^2 38° + \frac{1}{4}\sec^2 30°$$

$$= 1 + 0 + \frac{1}{4}\times\left(\frac{2}{\sqrt{3}}\right)^2$$

$$= 1 + \frac{1}{3}$$

$$= \frac{3+1}{3}$$

$$= \frac{4}{3}$$